

# **MAT205: Abstract Algebra II**

## **2. Group Actions**

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# What is a Group Action?

**Definition.** A (left) **group action** of  $G$  on a set  $X$  is a map  $G \times X \rightarrow X$ , written  $(g, x) \mapsto g \cdot x$ , satisfying:

1. **Identity:**  $e \cdot x = x$  for all  $x \in X$
2. **Compatibility:**  $(gh) \cdot x = g \cdot (h \cdot x)$  for all  $g, h \in G, x \in X$

Equivalently, a group action is a homomorphism  $\varphi : G \rightarrow \text{Sym}(X)$ .

Each  $g \in G$  defines a **bijection**  $x \mapsto g \cdot x$  on  $X$ .

## Key vocabulary:

- **Transitive:**  $\forall x, y \in X, \exists g \in G$  such that  $g \cdot x = y$  (only one orbit)
- **Faithful:** only  $e$  fixes every  $x$  ( $\varphi$  is injective)
- **Free:**  $g \cdot x = x$  for some  $x \Rightarrow g = e$  (no non-trivial fixed points)

# Group Actions $\leftrightarrow$ Homomorphisms

A group action  $G \curvearrowright X$  is **the same thing** as a homomorphism  $\varphi : G \rightarrow \text{Sym}(X)$ .

**Action  $\rightarrow$  homomorphism:** Given  $G \curvearrowright X$ , define  $\varphi(g)(x) = g \cdot x$ . Then  $\varphi(g)$  is a bijection  $X \rightarrow X$ , and  $\varphi(gh) = \varphi(g) \circ \varphi(h)$ .

**Homomorphism  $\rightarrow$  action:** Given  $\varphi : G \rightarrow \text{Sym}(X)$ , define  $g \cdot x = \varphi(g)(x)$ .

Under  $\varphi$ : **transitive**  $\leftrightarrow$  single orbit, **faithful**  $\leftrightarrow$   $\varphi$  injective, **free**  $\leftrightarrow$  no non-trivial fixed points.

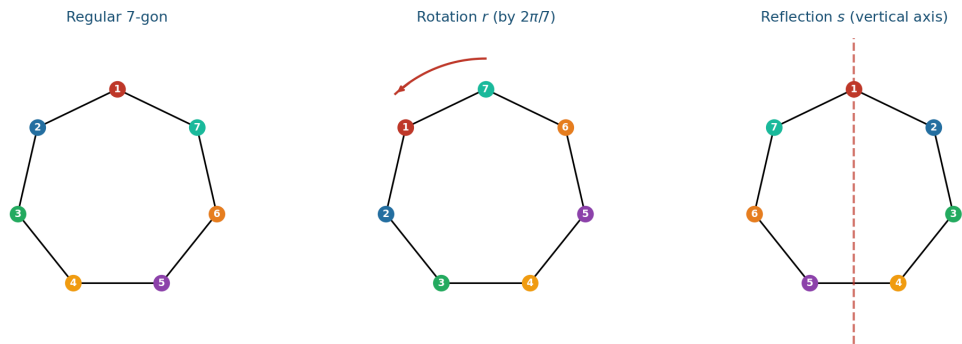
**Cayley's Theorem.** Every group  $G$  embeds into  $\text{Sym}(G)$  via the left-multiplication action. So every group is isomorphic to a subgroup of some symmetric group.



Cayley (1821-1895)

# Discrete Examples I: Symmetry of Polygons

**Example 1.** The dihedral group  $D_n$  acts on the vertices of a regular  $n$ -gon.



$D_7$  has  $2 \times 7 = 14$  elements: 7 rotations ( $e, r, r^2, \dots, r^6$ ) and 7 reflections.

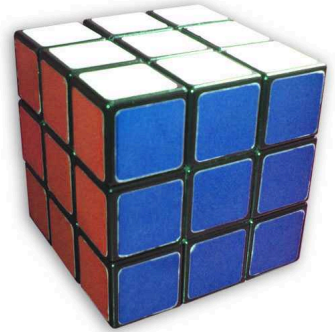
Each symmetry permutes the 7 vertices  $\implies D_7 \hookrightarrow S_7$ .

**Example 2.** The **Rubik's cube group** acts on the facelets of a Rubik's cube.

# The Rubik's Cube Group

In the language of group actions:

<b>Group <math>G</math></b>	$\langle U, D, L, R, F, B \rangle$ (generated by 6 face rotations)
<b>Set <math>X</math></b>	48 movable facelets (exclude 6 centers)
<b>Action</b>	$G \leq S_{48}$ , each move permutes facelets



**Faithful?** Yes – only  $e$  fixes all 48 facelets.

**Transitive?** No – two orbits: 24 corner facelets ( $8 \text{ cubies} \times 3$ ) and 24 edge facelets ( $12 \text{ cubies} \times 2$ ).

**Order:**  $|G| = \frac{8! \cdot 3^8 \cdot 12! \cdot 2^{12}}{12} \approx 4.3 \times 10^{19}$  (3 constraints: parity, twist, flip).

**God's number** = 20: diameter of Cayley graph (Rokicki et al., 2010).

# Discrete Examples II: Conjugation

**Example 3.** Any group  $G$  acts on **itself** by conjugation:

$$g \cdot x = gxg^{-1}$$

- **Orbit** of  $x$  = **conjugacy class**  $\text{Cl}(x) = \{gxg^{-1} \mid g \in G\}$
- **Stabilizer** of  $x$  = **centralizer**  $C_G(x) = \{g \in G \mid gx = xg\}$
- **Fixed points** = **center**  $Z(G) = \{x \in G \mid gx = xg \text{ for all } g\}$

This action is the key tool for Sylow theorems (next lectures).

# Conjugacy Classes of $GL_n(\mathbb{C})$

$A, B \in GL_n(\mathbb{C})$  are conjugate  $\iff$  they have the **same Jordan normal form** (up to block order).

A **Jordan block**:  $J_m(\lambda) = \begin{pmatrix} \lambda & & 1 \\ & \ddots & \\ & & \lambda \end{pmatrix} \in GL_m(\mathbb{C})$ , where  $\lambda \in \mathbb{C}^\times$ .

Every  $A$  is conjugate to  $J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$  (unique up to reordering).

**Conjugacy classes**  $\longleftrightarrow$  **Jordan types** (partitions of  $n$  labeled by nonzero eigenvalues):

$n$	Jordan type	Description
1	$(\lambda)$	one class per $\lambda \in \mathbb{C}^\times$
2	$\text{diag}(\lambda, \mu), \lambda \neq \mu$	diagonalizable, distinct eigenvalues
2	$\lambda I_2$	scalar matrix – <b>central</b> in $GL_2$
2	$J_2(\lambda)$	non-diagonalizable

# Discrete Examples III: Action on Cosets

**Example 4.**  $G$  acts on the coset space  $G/H$  by left multiplication:

$$g \cdot (aH) = (ga)H$$

This is always a **transitive** action (only one orbit).

**Stabilizer** of the coset  $eH$ :  $G_{eH} = H$ .

**Example 5.**  $S_n$  acts on polynomials by permuting variables:

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

The **fixed points** are the **symmetric polynomials** (e.g.,  $e_1 = \sum x_i$ ,  $e_2 = \sum_{i < j} x_i x_j$ ).

# Orbits

**Definition.** The **orbit** of  $x \in X$  under  $G$  is:

$$G \cdot x = \{g \cdot x \mid g \in G\}$$

**Theorem.** The orbits partition  $X$ :

$$X = \bigsqcup_{i \in I} G \cdot x_i$$

where  $\{x_i\}_{i \in I}$  is a set of orbit representatives (the index set  $I$  may be infinite).

*Proof.* Define  $x \sim y \iff \exists g \in G, g \cdot x = y$ . This is an equivalence relation:

- Reflexive:  $e \cdot x = x$
- Symmetric:  $g \cdot x = y \implies g^{-1} \cdot y = x$
- Transitive:  $g \cdot x = y, h \cdot y = z \implies (hg) \cdot x = z \square$

# Stabilizers

**Definition.** The **stabilizer** (or **isotropy group**) of  $x \in X$  is:

$$G_x = \{g \in G \mid g \cdot x = x\}$$

**Proposition.**  $G_x \leq G$  (it is a subgroup).

*Proof.* (i)  $e \cdot x = x$ , so  $e \in G_x$ . (ii) If  $g, h \in G_x$ , then  $(gh) \cdot x = g \cdot (h \cdot x) = g \cdot x = x$ . (iii) If  $g \in G_x$ , then  $g^{-1} \cdot x = g^{-1} \cdot (g \cdot x) = (g^{-1}g) \cdot x = x$ .  $\square$

Action	Stabilizer of $x$	Name
Conjugation on $G$	$C_G(x)$	centralizer
Left mult. on $G/H$	$H$	the subgroup itself
$SO(3)$ on $S^2$	rotations fixing $x \cong SO(2)$	–

# The Orbit-Stabilizer Theorem

**Theorem.** If  $G$  acts on  $X$  and  $x \in X$ , then

$$|G \cdot x| = [G : G_x] = \frac{|G|}{|G_x|}$$

*Proof sketch.* The map  $gG_x \mapsto g \cdot x$  is a well-defined bijection from  $G/G_x$  to  $G \cdot x$ .

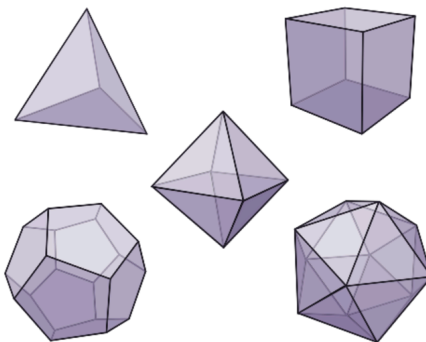
**Example:**  $D_4$  acts on vertices of a square.

- Orbit of vertex 1 = all 4 vertices (transitive), so  $|G \cdot 1| = 4$
- $|D_4| = 8$ , so  $|G_1| = 8/4 = 2$
- Indeed,  $G_1 = \{e, s\}$  where  $s$  is the reflection fixing vertex 1

**Corollary.** If  $G$  is finite and acts on  $X$ , then  $|G \cdot x|$  divides  $|G|$ .

# The Platonic Solids

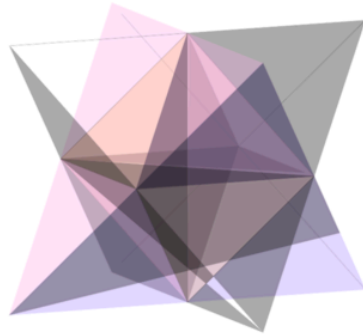
The five **Platonic solids** – the only convex regular polyhedra in  $\mathbb{R}^3$ .



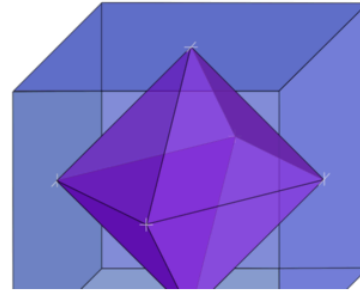
Dual pair	Rotation group	Full symmetry
Tetrahedron (self-dual)	$A_4$ (12)	$S_4$ (24)
Cube $\leftrightarrow$ Octahedron	$S_4$ (24)	$S_4 \times C_2$ (48)
Dodecahedron $\leftrightarrow$ Icosahedron	$A_5$ (60)	$A_5 \times C_2$ (120)

# Duality of Platonic Solids

The **dual** is formed by placing a vertex at each **face center**. Dual pairs share the same symmetry group.



Tetrahedron (self-dual)



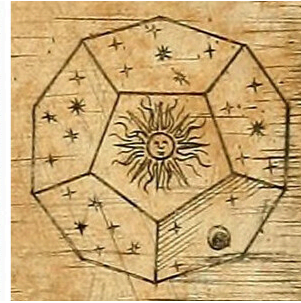
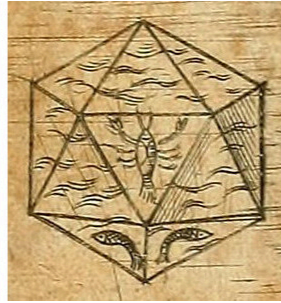
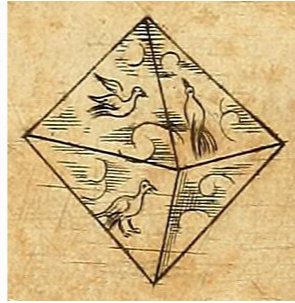
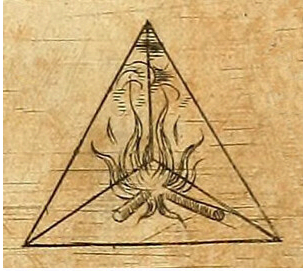
Cube  $\leftrightarrow$  Octahedron

- Tetrahedron  $\leftrightarrow$  Tetrahedron (self-dual:  $V = F = 4$ )
- Cube ( $V = 8, F = 6$ )  $\leftrightarrow$  Octahedron ( $V = 6, F = 8$ ): vertices  $\leftrightarrow$  faces
- Dodecahedron ( $V = 20, F = 12$ )  $\leftrightarrow$  Icosahedron ( $V = 12, F = 20$ )

$A_5 \cong$  rotation group of the icosahedron – the smallest **non-abelian simple group**. This connects to the **insolvability of the quintic** (Part II).

# Kepler's Harmonice Mundi (1619)

Kepler assigned each Platonic solid to a **classical element**:



Tetrahedron

Cube

Octahedron

Icosahedron

Dodecahedron

Fire

Earth

Air

Water

Cosmos

sharpest

most stable

"freely suspended"

most fluid

12 faces = zodiac

Kepler also used the Platonic solids to model **planetary orbits** in his *Mysterium Cosmographicum* (1596) – nesting each solid between planetary spheres. Though wrong as astronomy, it drove him toward his three laws of planetary motion.

# The Orthogonal and Special Orthogonal Groups

**Definition.** The **orthogonal group**  $O(n) = \{A \in GL_n(\mathbb{R}) \mid A^T A = I\}$  – the group of distance-preserving linear maps.

The **special orthogonal group**  $SO(n) = \{A \in O(n) \mid \det(A) = 1\}$  – the **rotation** group.

- $SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \mid \theta \in [0, 2\pi) \right\} \cong S^1$  – rotations of the plane
- $SO(3)$  – rotations of  $\mathbb{R}^3$ . Every element is a rotation about some axis (Euler's rotation theorem).
- $O(n) = SO(n) \sqcup \{\text{reflections}\}$ :  $[O(n) : SO(n)] = 2$

# Continuous Examples I: $SO(3)$ on $S^2$

**Example 6.**  $SO(3)$  acts on  $S^2$  – **transitively** (one orbit = entire sphere).

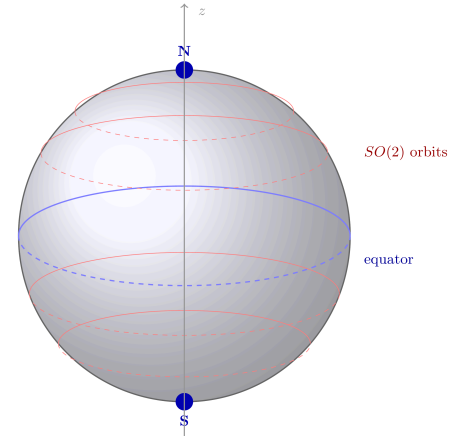
- **Stabilizer** of north pole  $N$ : rotations about  $z$ -axis =  $SO(2)$
- **Orbit-stabilizer:**  $S^2 \cong SO(3)/SO(2)$

The subgroup  $SO(2) \leq SO(3)$  acts on  $S^2$  **non-transitively**:

- **Orbits** = latitude circles (red in figure)
- **Fixed points** = north and south poles

**Example 7.**  $O(n) \curvearrowright \mathbb{R}^n$  (orthogonal transformations).

- **Orbits** = concentric spheres  $S_r^{n-1}$
- **Stabilizer** of  $(r, 0, \dots, 0) \cong O(n-1)$



$SO(3)$ : transitive (one orbit =  $S^2$ )

$SO(2)$ : orbits = latitude circles

# Continuous Examples II: Linear and Modular

**Example 8.**  $GL_n(\mathbb{R})$  acts on  $\mathbb{R}^n$  by matrix multiplication:  $A \cdot v = Av$ .

- **Orbits:**  $\{0\}$  and  $\mathbb{R}^n \setminus \{0\}$  (any nonzero vector maps to any other)
- **Stabilizer** of  $e_1 = (1, 0, \dots, 0)^T$ : matrices with first column  $e_1$

**Example 9.**  $SL_2(\mathbb{Z})$  acts on the upper half-plane  $\mathbb{H}$  (from Lecture 1):

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}$$

- **Orbit space**  $\mathbb{H}/SL_2(\mathbb{Z}) =$  modular curve (parameterizes elliptic curves)
- **Stabilizer** of  $z = i$ :  $\{\pm I, \pm \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\} \cong \mathbb{Z}/4\mathbb{Z}$  (in  $PSL_2(\mathbb{Z})$ :  $\mathbb{Z}/2\mathbb{Z}$ )

# Klein's Erlangen Program (1872)

Felix Klein proposed: **a geometry is a group action.**

A **geometry** = a set  $X$  together with a group  $G$  acting on  $X$ . The "geometric properties" of  $X$  are exactly the properties invariant under  $G$ .

Geometry	Space $X$	Group $G$	Invariants
Euclidean	$\mathbb{R}^n$	$O(n) \ltimes \mathbb{R}^n$	distances, angles
Affine	$\mathbb{R}^n$	$GL_n \ltimes \mathbb{R}^n$	parallelism, ratios
Projective	$\mathbb{P}^n$	$PGL_{n+1}$	cross-ratio, collinearity
Hyperbolic	$\mathbb{H}^n$	$SO(n, 1)$	hyperbolic distance

**"More symmetry = fewer invariants = coarser geometry."**

This viewpoint unifies all of classical geometry under group theory – and motivates everything in this course.

# The Class Equation

Apply conjugation action  $G \curvearrowright G$  with  $g \cdot x = gxg^{-1}$ :

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

where the sum runs over representatives of non-central conjugacy classes.

**Example:**  $S_3$  has order 6, center  $Z(S_3) = \{e\}$ .

Conjugacy class	Size	$[G : C_G(x)]$
$\{e\}$	1	(center)
$\{(12), (13), (23)\}$	3	$6/2 = 3$
$\{(123), (132)\}$	2	$6/3 = 2$

# Application: Center of a $p$ -Group

**Theorem.** If  $|G| = p^n$  with  $p$  prime and  $n \geq 1$ , then  $Z(G) \neq \{e\}$ .

*Proof.* Apply the **class equation**:

$$|G| = |Z(G)| + \sum_i [G : C_G(x_i)]$$

Each  $[G : C_G(x_i)]$  divides  $|G| = p^n$  and is  $> 1$ , so  $p \mid [G : C_G(x_i)]$ .

Therefore  $p \mid |G| - \sum [G : C_G(x_i)] = |Z(G)|$ .

Since  $e \in Z(G)$ , we have  $|Z(G)| \geq p > 1$ .  $\square$

**Corollary.** Groups of order  $p^2$  are abelian.

**Homework.** Prove: if  $G/Z(G)$  is cyclic, then  $G$  is abelian. Use this to prove the corollary.

# Burnside's Lemma

**Theorem** (Burnside / Cauchy-Frobenius). The number of orbits of  $G$  acting on  $X$  is:

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

where  $X^g = \{x \in X \mid g \cdot x = x\}$  is the set of fixed points of  $g$ .

*Proof.* Count the set  $S = \{(g, x) \in G \times X \mid g \cdot x = x\}$  in two ways:

- By  $g$ :  $|S| = \sum_{g \in G} |X^g|$
- By  $x$ :  $|S| = \sum_{x \in X} |G_x| = \sum_{\text{orbits } O} \sum_{x \in O} \frac{|G|}{|O|} = |X/G| \cdot |G| \quad \square$

**Q:** How many distinct ways to color the vertices of a square with 2 colors, up to rotation and reflection?

Distinct colorings of a square (2 colors, up to  $D_4$ )



# Burnside: Square Coloring

Vertices 1, 2, 3, 4 clockwise.  $D_4 = \langle r, s \rangle$ ,  $r = (1234)$ ,  $s = (24)$ . Color each vertex B or W.  $|X^g| = 2^{(\text{number of cycles})}$ .

$g$	Permutation	Cycles	$ X^g $
$e$	id	(1)(2)(3)(4) – 4 cycles	16
$r, r^3$	(1234), (1432)	1 cycle each	2, 2
$r^2$	(13)(24)	2 cycles	4
$s$	(24)	(1)(3)(24) – 3 cycles	8
$r^2s$	(13)	(2)(4)(13) – 3 cycles	8
$rs, r^3s$	(12)(34), (14)(23)	2 cycles each	4, 4

$$|X/G| = \frac{1}{8}(16 + 2 + 4 + 2 + 8 + 4 + 8 + 4) = \frac{48}{8} = 6$$

# Homework (Lecture 2)

## Problem 1. Bracelets with 7 beads

Let  $X$  be the set of all colorings of the vertices of a regular heptagon using two colors. Let  $D_7$  act on  $X$  by permuting the vertices.

- Compute  $|X|$ .
- How many colorings are fixed by the identity element?
- How many colorings are fixed by a nontrivial rotation?
- How many colorings are fixed by a reflection?
- Use Burnside's lemma to compute the number of distinct bracelets.

## Problem 2. The sphere as a homogeneous space

- Prove that  $SO(3)$  acts on the unit sphere  $S^2 \subset \mathbb{R}^3$ .
- Prove that this action is transitive.
- Let  $N = (0, 0, 1)$  be the north pole. Compute the stabilizer  $\text{Stab}(N)$ .

# Looking Ahead

We now have the tools for **Sylow theory**:

- **Lecture 3**: Cauchy's theorem (via group action on tuples) and the normalizer lemma
- **Lecture 4**: Sylow I (existence), Sylow II (conjugacy), Sylow III (counting)

The key idea: choose the right group action on the right set, then count.

**"Group actions are the single most important tool in finite group theory."**

**Questions?**