

MAT205 Lecture 2 Homework

Problem 1. Bracelets with 7 Beads

Let X be the set of all colorings of the vertices of a regular heptagon using two colors. Let D_7 act on X by permuting the vertices.

- Compute $|X|$.
- How many colorings are fixed by the identity element?
- How many colorings are fixed by a nontrivial rotation?
- How many colorings are fixed by a reflection?
- Use Burnside's lemma to compute the number of distinct bracelets.

Problem 2. The Sphere as a Homogeneous Space

- Prove that $SO(3)$ acts on the unit sphere $S^2 \subset \mathbb{R}^3$.
- Prove that this action is transitive.
- Let $N = (0, 0, 1)$ be the north pole. Compute the stabilizer $\text{Stab}(N)$.
- Prove that $\text{Stab}(N) \cong SO(2)$.
- Conclude that

$$S^2 \cong SO(3)/SO(2).$$

- Define a map

$$\Phi : SO(3)/SO(2) \rightarrow S^2, \quad \Phi(gSO(2)) = g \cdot N.$$

Prove that Φ is well-defined and bijective.