MAT205 Lecture 3 Homework

Problem 1. The Rotation Group of the Tetrahedron

Let G be the rotation group of a regular tetrahedron.

a. Show that each rotation permutes the four faces, so there is a homomorphism

$$\varphi : G \to S_4.$$

b. Prove that φ is injective.

Hint: if a rotation fixes all four faces, then it fixes the whole tetrahedron.

c. Show that $|G| = 12$.

Hint: count rotations by axis type: identity, 120° and 240° rotations, and 180° rotations.

d. Conclude that $G \cong A_4$.

Hint: an injective homomorphism from a group of order 12 into S_4 must land in the subgroup of even permutations.

e. Find the numbers of Sylow 2-subgroups and Sylow 3-subgroups of A_4.

f. Explain geometrically what these Sylow subgroups correspond to.

Problem 2. The Rotation Group of the Cube

Let G be the rotation group of a cube.

a. Show that each rotation permutes the four body diagonals, so there is a homomorphism

$$\psi : G \to S_4.$$

b. Prove that ψ is injective.

Hint: if a rotation fixes all four body diagonals, then it fixes the cube.

c. Show that $|G| = 24$.

Hint: count rotations by axis type.

d. Conclude that $G \cong S_4$.

e. Use Sylow theory to determine the numbers of Sylow 2-subgroups and Sylow 3-subgroups of G.

f. Describe geometrically one Sylow 3-subgroup and one Sylow 2-subgroup.