MAT205 Lecture 4 Homework

Fraleigh-inspired practice on applications of the Sylow theorems and semidirect products.

Problem 1. Groups of Order 21

Classify all groups of order 21 up to isomorphism.

Hint: apply Sylow's theorems to determine the number of Sylow 3- and 7-subgroups, then analyze the possible actions.

Problem 2. No Group of Order 20 Is Simple

Show that every group of order 20 has a normal Sylow 5-subgroup. Conclude that no group of order 20 is simple.

Hint: count Sylow 5-subgroups using the divisibility and congruence conditions.

Problem 3. No Group of Order 30 Is Simple

Show that every group of order 30 contains a normal subgroup of order 5 or 3. Conclude that no group of order 30 is simple.

Hint: count elements of orders 5 and 3, and use the resulting inequalities.

Problem 4. S_3 as a Semidirect Product

Show that

$$S_3 \;\cong\; \mathbb{Z}_3 \rtimes \mathbb{Z}_2,$$

and identify explicitly the homomorphism

$$\varphi : \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}_3).$$

Hint: let $\mathbb{Z}_3 = \langle r \rangle$ sit inside S_3 as the normal subgroup of 3-cycles, and $\mathbb{Z}_2 = \langle s \rangle$ sit as any transposition. Describe how conjugation by s acts on r.