MAT205 Lecture 5 Homework

Practice on free abelian groups, the classification of finitely generated abelian groups, and its invariants.

Problem 1. Torsion Is a Subgroup

Let A be an abelian group, and define

$$T(A) \;=\; \{\, a \in A : n\,a = 0 \text{ for some integer } n \neq 0 \,\}.$$

a. Show that $T(A)$ is a subgroup of A.

Hint: check the three subgroup axioms. For closure, if $na = 0$ and $mb = 0$ with $n, m \neq 0$, find a single nonzero integer killing $a + b$.

b. Show that $A/T(A)$ is **torsion-free**: i.e., $T(A/T(A)) = 0$.

c. Give an example of a *non-abelian* group in which the set of torsion elements is **not** a subgroup.

Hint: try the infinite dihedral group $D_\infty = \langle r, s \mid s^2 = 1, srs = r^{-1} \rangle$.

Problem 2. Determinantal Test for a Basis of $\mathbb{Z}^n$

Let $x_1, \dots, x_n \in \mathbb{Z}^n$ and let M be the $n \times n$ integer matrix whose columns are $x_1, \dots, x_n$.

a. Prove that $\{x_1, \dots, x_n\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ if and only if $\det M \in \{+1, -1\}$.

Hint: "basis" means every element of $\mathbb{Z}^n$ is uniquely an integer linear combination of the x_i. Equivalently, M should be invertible with an integer-matrix inverse.

b. Decide whether each of the following is a basis of $\mathbb{Z}^2$:

$$\{(2,1),\ (1,2)\}, \qquad \{(3,1),\ (1,2)\}, \qquad \{(3,6),\ (1,2)\}.$$

Problem 3. Abelian Groups of Order 72

List all abelian groups of order 72 up to isomorphism.

For each one, give both its invariant-factor decomposition

$$\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \cdots \oplus \mathbb{Z}/d_k, \qquad d_1 \mid d_2 \mid \cdots \mid d_k,$$

and its primary (prime-power) decomposition.

Hint: $72 = 2^3 \cdot 3^2$. Work prime by prime using partitions of 3 and partitions of 2.

Problem 4. Free Rank Is an Invariant

Prove that $\mathbb{Z}^m \not\cong \mathbb{Z}^n \oplus T$ whenever $m > n$ and T is a finite (torsion) abelian group.

Hint: quotient by the torsion subgroup (Problem 1) to reduce to showing $\mathbb{Z}^m \not\cong \mathbb{Z}^n$ for $m \neq n$. For that, look at $G/2G$: if $G = \mathbb{Z}^k$, then $G/2G \cong (\mathbb{Z}/2)^k$, a group of order 2^k — so its size reads off k.

Problem 5. Counting d-Torsion

Let $G \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k$ be a finite abelian group in invariant-factor form, and for $d \geq 1$ set

$$G[d] \;=\; \{g \in G : d\,g = 0\}.$$

a. Prove that

$$|G[d]| \;=\; \prod_{i=1}^{k} \gcd(d, d_i).$$

Hint: work component by component and use the fact that in $\mathbb{Z}/m$, the elements killed by d form the subgroup of order $\gcd(d, m)$.

b. Use the formula to count the number of elements of order exactly 4 in

$$\mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8.$$

Hint: elements of order exactly 4 are those in $G[4]$ but not in $G[2]$.