MAT205 Lecture 6 Homework

Practice on the classification of finitely generated abelian groups and its applications.

Problem 1. Abelian Groups of Order 72

List all abelian groups of order 72 up to isomorphism.

For each one, give both its invariant-factor decomposition

$$\mathbb{Z}/d_1 \oplus \mathbb{Z}/d_2 \oplus \cdots \oplus \mathbb{Z}/d_k, \qquad d_1 \mid d_2 \mid \cdots \mid d_k,$$

and its primary (prime-power) decomposition.

Hint: $72 = 2^3 \cdot 3^2$. Work prime by prime, using partitions of 3 and partitions of 2.

Problem 2. Counting d-Torsion

Let $G \cong \mathbb{Z}/d_1 \oplus \cdots \oplus \mathbb{Z}/d_k$ be a finite abelian group in invariant-factor form, and for $d \geq 1$ set

$$T_d(G) \;=\; [G]_d \;:=\; \{g \in G : d\,g = 0\}.$$

a. Prove that

$$|T_d(G)| \;=\; |[G]_d| \;=\; \prod_{i=1}^{k} \gcd(d, d_i).$$

Hint: work component by component. In $\mathbb{Z}/m$, the elements killed by d form the subgroup of order $\gcd(d, m)$.

b. Use the formula to count the number of elements of order exactly 4 in

$$\mathbb{Z}/4 \oplus \mathbb{Z}/2 \oplus \mathbb{Z}/8.$$

Hint: elements of order exactly 4 are those in $T_4(G) = [G]_4$ but not in $T_2(G) = [G]_2$.

Problem 3. The Unit Group $(\mathbb{Z}/105\mathbb{Z})^\times$

Determine the structure of $(\mathbb{Z}/105\mathbb{Z})^\times$ as an abelian group — in both invariant-factor and primary-decomposition form.

Hint: $105 = 3 \cdot 5 \cdot 7$. Use the Chinese Remainder Theorem to split into $(\mathbb{Z}/p)^\times$ pieces, then apply the theorem that finite subgroups of a field are cyclic.