MAT205: Abstract Algebra II

Free Groups: Groups Before Relations

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Goal of This Lecture

We have just seen that finitely generated abelian groups are controlled by generators and integer-linear relations.

Today we remove the word **abelian**.

$$\boxed{\text{What is the freest group generated by a set } S\text{?}}$$

The answer is the **free group** $F(S)$.

Its guiding principle:

$$\boxed{\text{A map out of } F(S) \text{ is determined by arbitrary images of } S.}$$

Free, But Bound to a Basis

Bob Dylan is an American singer-songwriter and the 2016 Nobel laureate in Literature.

In **Ballad in Plain D** (1964), Dylan asks:

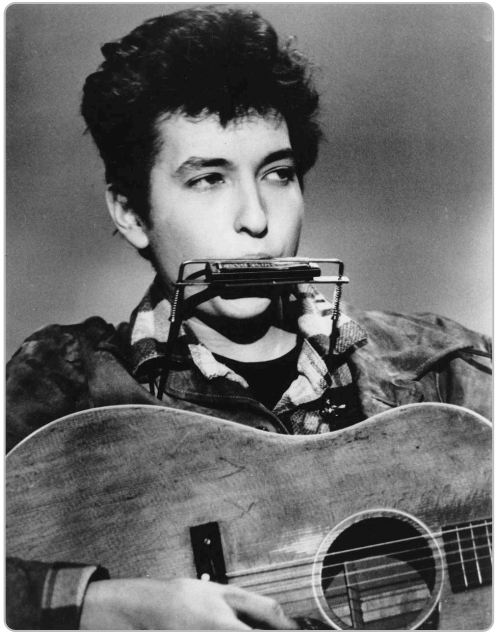
> "Are birds free from the chains of the skyway?"

For us, the point is:

$$\boxed{\text{A free group is bound to its basis } S.}$$

The basis gives the coordinates.

Freedom means: after choosing images of S, there are **no extra relation checks**.



Bob Dylan, 1963. Wikimedia Commons, public domain.

Sources: Bob Dylan official lyrics; Nobel Prize; photo

Part I

Universal Property First

Recall: Free Abelian Groups

Let S be a set. The free abelian group on S, denoted $F^{ab}(S)$, satisfies:

$$\mathrm{Hom}_{\mathbf{Ab}}(F^{ab}(S), A) \cong \mathrm{Func}(S, A)$$

for every abelian group A.

Meaning:

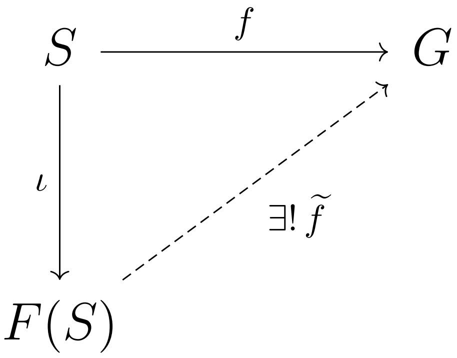
> To define a homomorphism $F^{ab}(S) \to A$, it is enough to choose where each generator $s \in S$ goes.

But the target A must be abelian.

Definition: Free Group

Definition. A group $F(S)$, together with $\iota : S \to F(S)$, is the **free group on S** if:

For every group G and every function $f : S \to G$, there is a unique homomorphism $\tilde{f} : F(S) \to G$ making this triangle commute:



Reading the Diagram

Commutative means:

$$\tilde{f}(\iota(s)) = f(s) \qquad \text{for all } s \in S.$$

Equivalently:

$$\mathrm{Hom}_{\mathbf{Grp}}(F(S), G) \cong \mathrm{Func}(S, G).$$

The Key Difference

Object	Universal property	Target groups
Free abelian group $F^{ab}(S)$	maps determined by S	abelian groups
Free group $F(S)$	maps determined by S	all groups

So $F(S)$ must avoid imposing any relation that arbitrary target groups may fail to satisfy.

Slogan:

$$\boxed{\text{Free group} = \text{basis } S \text{ with no extra relations.}}$$

Why $F(a, b)$ Cannot Be Abelian

Suppose $ab = ba$ in $F(a, b)$.

Take any group G and any two elements $g, h \in G$. By the universal property, there is a homomorphism

$$\tilde{f} : F(a, b) \to G, \qquad a \mapsto g, \quad b \mapsto h.$$

Then

$$gh = \tilde{f}(ab) = \tilde{f}(ba) = hg.$$

So every group would be abelian.

But in S_3,

$$(12)(123) \neq (123)(12).$$

Therefore $ab \neq ba$ in $F(a, b)$.

First Examples

No generators.

$$F(\varnothing) \cong \{1\}.$$

There is exactly one homomorphism from the trivial group to any group.

One generator.

$$F(\{x\}) \cong \mathbb{Z}.$$

A homomorphism out of $\mathbb{Z}$ is determined by the image of 1.

Two generators.

$$F(a, b)$$

is the first genuinely non-abelian free group.

When Checks Become Necessary

Compare with a group given by a relation:

$$\mathbb{Z}_n = \langle x \mid x^n = 1 \rangle.$$

To define a homomorphism $\mathbb{Z}_n \to G$, choosing $x \mapsto g$ is not enough.

We must check:

$$g^n = 1.$$

This is the key role of a relation:

$$\boxed{\text{Relations are exactly the conditions maps must respect.}}$$

Part II

The Word Model

Words in Generators

Let $S = \{a, b\}$.

We introduce formal inverse symbols:

$$a^{-1}, \qquad b^{-1}.$$

A **word** is a finite string such as:

$$ab^{-1}a^{-1}b, \qquad a^2b^{-3}ab, \qquad b^{-1}a^{-1}bab.$$

At this level, words are just strings. They are not yet simplified.

Cancellation

The only forced simplifications are:

$$aa^{-1} \rightsquigarrow 1, \qquad a^{-1}a \rightsquigarrow 1,$$

and similarly for every generator.

Example:

$$abb^{-1}a^{-1} \rightsquigarrow aa^{-1} \rightsquigarrow 1.$$

But

$$aba^{-1}b^{-1}$$

does not simplify. In a free group, this is not the identity.

Reduced Words

Definition. A word is **reduced** if it contains no adjacent pair of the form

$$ss^{-1} \quad \text{or} \quad s^{-1}s.$$

Examples of reduced words:

$$ab, \qquad ba, \qquad aba^{-1}b^{-1}, \qquad a^3b^{-2}ab.$$

Examples not reduced:

$$abb^{-1}a, \qquad a^{-1}ab, \qquad ba^{-1}ab^{-1}b.$$

Normal Form Theorem

Theorem. Every element of $F(S)$ has a unique reduced word representative.

This lets us decide equality in a free group:

$$u = v \quad \Longleftrightarrow \quad \mathrm{red}(u) = \mathrm{red}(v).$$

Consequences:

$$ab \neq ba, \qquad aba^{-1}b^{-1} \neq 1.$$

This is the free group's **word problem**, and here it is easy: reduce the word.

Multiplication

Multiplication in $F(S)$:

$$\text{multiply two reduced words} = \text{concatenate, then reduce.}$$

Example:

$$(ab^{-1}a)(a^{-1}ba^{-1})$$

First concatenate:

$$ab^{-1}aa^{-1}ba^{-1}$$

Then reduce:

$$ab^{-1}ba^{-1}.$$

Inverses

To invert a word:

1. reverse the order;
2. invert every letter.

Example:

$$(ab^{-1}a^{-1}b)^{-1} = b^{-1}aba^{-1}.$$

Then:

$$(ab^{-1}a^{-1}b)(b^{-1}aba^{-1}) = 1$$

after cancellation.

The Word Model Is Subtle

It is tempting to say:

> $F(S)$ is the set of reduced words.

But to make that a rigorous construction, we must prove:

- reduction terminates;
- different reduction orders give the same final reduced word;
- multiplication is associative;
- inverses behave correctly.

This is why a formal system like Mathlib first constructs a quotient, then proves a normal form theorem.

Part III

Abelianization and Rank

Commutators

For elements $g, h \in G$, the **commutator** is:

$$[g, h] = ghg^{-1}h^{-1}.$$

It measures failure to commute:

$$[g, h] = 1 \quad \Longleftrightarrow \quad gh = hg.$$

The **commutator subgroup** is:

$$[G, G] = \langle [g, h] \mid g, h \in G \rangle.$$

Abelianization

Definition. The **abelianization** of G is

$$G^{ab} = G/[G,G].$$

In G^{ab}, all commutators become trivial:

$$[g,h] \mapsto 1.$$

So G^{ab} is abelian.

Slogan:

$$\boxed{G^{ab} \text{ is the best abelian approximation of } G.}$$

Universal Property of Abelianization

Let A be an abelian group.

Every homomorphism

$$\varphi : G \to A$$

kills all commutators:

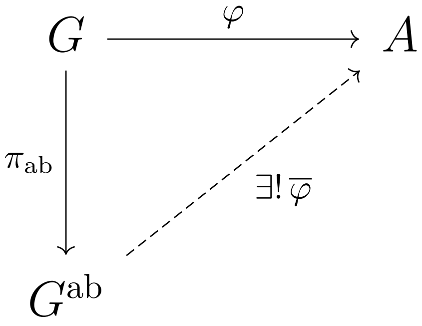
$$\varphi([g,h]) = \varphi(g)\varphi(h)\varphi(g)^{-1}\varphi(h)^{-1} = 1.$$

Therefore $[G,G] \subseteq \ker \varphi$, so φ factors uniquely:

$$G \longrightarrow G^{ab} \longrightarrow A.$$

Abelianization as a Universal Property

Let A be an abelian group and let $\varphi : G \to A$ be any homomorphism.



The diagram means:

- $\pi_{\mathrm{ab}} : G \to G^{ab}$ is the quotient map;
- every map $\varphi : G \to A$ factors through G^{ab};
- the factorization $\overline{\varphi}$ is unique.

$$\mathrm{Hom}_{\mathbf{Ab}}(G^{ab}, A) \cong \mathrm{Hom}_{\mathbf{Grp}}(G, A)$$

This is the same style of idea as free groups:

- free group controls arbitrary maps from generators;
- abelianization controls maps from a group into abelian targets.

Examples of Abelianization

If G is already abelian:

$$G^{ab} \cong G.$$

For S_3:

$$S_3^{ab} \cong \mathbb{Z}_2.$$

The sign map $S_3 \to \{\pm 1\}$ is the universal map from S_3 to an abelian group.

For a free group:

$$F(a,b)^{ab} \cong \mathbb{Z}^2.$$

The words remember only the total exponent of a and the total exponent of b.

Free Group to Free Abelian Group

Let S be any set.

Claim:

$$\boxed{F(S)^{ab} \cong F^{ab}(S).}$$

Proof by universal property:

For every abelian group A,

$$\mathrm{Hom}_{\mathbf{Ab}}(F(S)^{ab}, A) \cong \mathrm{Hom}_{\mathbf{Grp}}(F(S), A) \cong \mathrm{Func}(S, A).$$

This is exactly the universal property of the free abelian group on S.

Concretely, $F^{ab}(S) \cong \mathbb{Z}[S]$.

Rank Is a Cardinality

For a free group $F(S)$, its **rank** means the cardinality of a basis:

$$\operatorname{rank} F(S) = |S|.$$

So the theorem says:

$$F(S) \cong F(T) \quad \Longrightarrow \quad |S| = |T|.$$

Rank of Free Groups Is Well-Defined

Suppose

$$F(S) \cong F(T).$$

Abelianize both sides:

$$F(S)^{ab} \cong F(T)^{ab}.$$

But

$$F(S)^{ab} \cong \mathbb{Z}[S], \qquad F(T)^{ab} \cong \mathbb{Z}[T].$$

Rank of free abelian groups is well-defined, so

$$|S| = |T|.$$

Therefore the rank of a free group is well-defined.

Part IV

A Short Mathlib View

Mathlib: The Definition Looks Tiny

The final type definition is only two lines:

```
def FreeGroup (α : Type u) : Type u :=
  Quot ◁ @FreeGroup.Red.Step α
```

But this line depends on a reduction relation:

```
inductive FreeGroup.Red.Step :
  List (α × Bool) → List (α × Bool) → Prop
```

The point for students:

$$\boxed{\text{the definition is short because the proof infrastructure is large.}}$$

Source: Mathlib `Red.Step`, definition of `FreeGroup`

How Much Infrastructure?

At this Mathlib commit:

Before the definition

- `Red.Step` : line 77.
- `FreeGroup` : line 472.
- About 400 lines prepare the quotient model.

For normal forms

- `Reduce.lean` : 404 lines.
- `reduce` : line 39.
- `toWord` : line 191.
- Core theorems run through line 240.

So the classroom slogan

$$\text{“cancel } ss^{-1} \text{ until reduced”}$$

corresponds to hundreds of lines of formal mathematics.

Sources: Mathlib `Basic.lean` , Mathlib `Reduce.lean`

What Has To Be Proved?

To make words into a group, Mathlib proves that:

- cancellation is allowed inside longer words;
- reduction is compatible with the quotient;
- different reduction orders lead to the same result;
- multiplication by concatenation is well-defined;
- the empty reduced word is exactly the identity.

Only after this work can Mathlib expose the friendly API:

```
FreeGroup.of   : α → FreeGroup α
FreeGroup.lift : (α → β) ≃ (FreeGroup α →* β)
```

Sources: Mathlib reduction lemmas, Mathlib universal property

Normal Form Is a Theorem

After constructing the quotient, Mathlib later proves the reduced-word theorem:

```
FreeGroup.toWord : FreeGroup α → List (α × Bool)
```

Important theorems:

```
toWord_injective
toWord_mul
toWord_eq_nil_iff
```

Translation:

- `toWord_injective` : reduced form uniquely determines the group element.
- `toWord_mul` : multiplication is concatenate, then reduce.
- `toWord_eq_nil_iff` : the empty reduced word is the identity.

Source: Mathlib reduced-word API

Part V

Group Presentations

Definition: Group Presentation

Let S be a set of generators, and let $R \subset F(S)$ be a set of words.

The group presented by S and R is

$$\langle S \mid R \rangle = F(S)/\langle\langle R \rangle\rangle.$$

Here $\langle\langle R \rangle\rangle$ is the smallest normal subgroup of $F(S)$ containing all words in R.

Read this as:

$$\boxed{\text{generators } S, \quad \text{relations } r = 1 \text{ for every } r \in R.}$$

Maps Out of a Presentation

To define a homomorphism

$$\varphi : \langle S \mid R \rangle \to G,$$

start with a function $f : S \to G$.

1. Extend it uniquely to $\tilde{f} : F(S) \to G$.
2. Check every relator:

$$\tilde{f}(r) = 1 \qquad \text{for every } r \in R.$$

If the checks pass, $\tilde{f}$ factors uniquely through

$$F(S)/\langle\langle R \rangle\rangle.$$

So:

First Presentations

Free group.

$$F(a, b) = \langle a, b \mid \rangle.$$

There are no relation checks.

Infinite cyclic group.

$$\mathbb{Z} = \langle x \mid \rangle.$$

One generator, no finite-order relation.

Cyclic group of order n.

$$\mathbb{Z}_n = \langle x \mid x^n \rangle.$$

The relation means $x^n = 1$.

Commutativity as a Relation

The relation $ab = ba$ is written as

$$aba^{-1}b^{-1} = 1.$$

Therefore

$$\mathbb{Z}^2 \cong \langle a, b \mid aba^{-1}b^{-1} \rangle.$$

Compare:

$$F(a,b) = \langle a, b \mid \rangle, \qquad \mathbb{Z}^2 = \langle a, b \mid [a,b] \rangle.$$

Adding one relation turns the free group on two generators into the free abelian group on two generators.

A Non-Abelian Example: S_3

Let $r = (123)$ and $s = (12)$ in S_3.

Then:

$$r^3 = 1, \qquad s^2 = 1, \qquad srs = r^{-1}.$$

A presentation is:

$$S_3 \cong \langle r, s \mid r^3,\ s^2,\ srsr \rangle.$$

Read the last relation as:

$$srs = r^{-1}.$$

This presentation says: one rotation, one reflection, and the rule for how they interact.

Dihedral Groups D_n

Convention in this lecture:

$$D_n = \{\text{symmetries of a regular } n\text{-gon}\}, \qquad |D_n| = 2n.$$

Let r be rotation by $2\pi/n$, and let s be a reflection.

$$r^n = 1, \qquad s^2 = 1, \qquad srs = r^{-1}.$$

So:

$$D_n \cong \langle r, s \mid r^n,\ s^2,\ srsr \rangle.$$

For $n = 3$, this recovers $D_3 \cong S_3$.

Coxeter Presentations from Diagrams

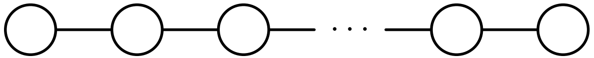
A Coxeter diagram is a compact way to write a presentation.

- Each vertex gives a generator s_i.
- Every generator is an involution: $s_i^2 = 1$.
- An edge labelled m means $(s_i s_j)^m = 1$.
- An unlabeled edge means $m = 3$.
- No edge means $m = 2$, so s_i and s_j commute.

The diagram is not decoration: it is the presentation.

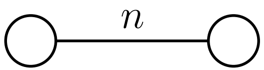
The First Diagrams: A_n and $I_2(n)$

A_n



n vertices; unlabeled edges mean $m = 3$

$I_2(n)$



two reflections; the product has order n

TikZ source: `slides/diagrams/coxeter-a-i.tex`

For type A_n:

$$W(A_n) \cong S_{n+1}.$$

For type $I_2(n)$:

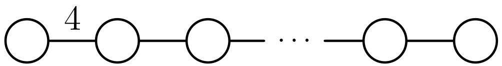
$$I_2(n) = \langle s, t \mid s^2,\ t^2,\ (st)^n \rangle.$$

Geometrically, s, t are two reflections and st is rotation by $2\pi/n$.

Thus $I_2(n) \cong D_n$.

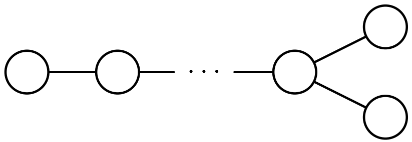
Classical Types: $B_n = C_n$ and D_n

$B_n = C_n$



one edge labelled 4; the rest are unlabeled

D_n



a simply-laced chain with a fork at one end

TikZ source: `slides/diagrams/coxeter-bcd.tex`

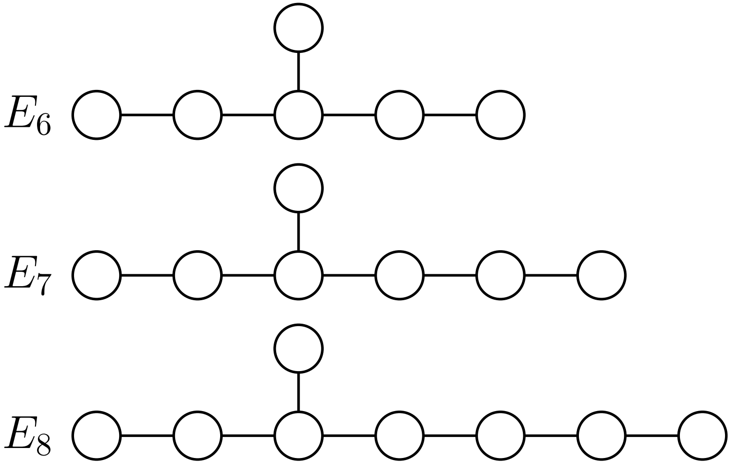
The Coxeter diagram records only the reflection-group presentation.

For this reason, B_n and C_n have the same Coxeter diagram:

$$W(B_n) \cong W(C_n).$$

Type D_n is simply-laced: all displayed edges have $m = 3$.

Exceptional Type E



exceptional simply-laced finite Coxeter diagrams

TikZ source: `slides/diagrams/coxeter-e.tex`

These are the exceptional finite simply-laced diagrams:

$$E_6, \qquad E_7, \qquad E_8.$$

They do not come from the infinite families A_n, B_n, C_n, D_n.

This is why Coxeter diagrams are useful: the picture is a classification language.

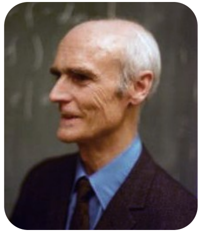
H. S. M. Coxeter (1907–2003)

Coxeter was a British-Canadian geometer, famous for regular polytopes, reflection groups, and the systematic use of Coxeter diagrams.

For this lecture, the key idea is:

$$\boxed{\text{many symmetry groups are understood through presentations.}}$$

Coxeter groups are the cleanest example: generators are reflections, and the relations record pairwise angles.



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Part VI

Graphs and Free Groups

Paths as Words

A graph path behaves like a word.

If an oriented edge is called a, traversing it backward is a^{-1}.

Immediate backtracking:

$$aa^{-1}$$

is the same as going out along an edge and immediately coming back.

So word reduction is graph path simplification:

$$\boxed{\text{cancel adjacent inverse letters} = \text{remove immediate backtracking.}}$$

Bouquet of S^1

Let

$$X = \bigvee^{n} S^1$$

be n circles joined at one common basepoint.

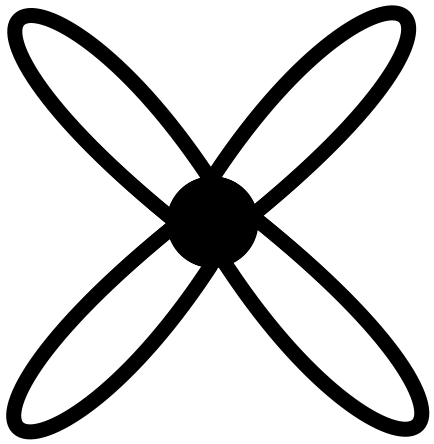


Image: Michel Bakni, Wikimedia Commons, CC BY-SA 4.0

Then:

$$\pi_1(X) \cong F_n.$$

Fundamental Group of a Graph

Theorem. If Γ is a connected graph, then $\pi_1(\Gamma)$ is a free group.

If Γ is finite, then more precisely:

$$\pi_1(\Gamma) \cong F_r.$$

Here r is the number of independent cycles in the graph.

Graphs Make Words Visible

The graph model turns abstract group theory into geometry:

- words are paths;
- inverse letters are reversed edges;
- reduction removes backtracking;
- free generators are independent cycles;
- relations create loops that are declared trivial.

This is the cleanest intuition for why free groups have no hidden relations.

Part VII

Subgroups of Free Groups

Nielsen-Schreier Theorem

Theorem (Nielsen-Schreier).

Every subgroup of a free group is free.

$$H \leq F(S) \quad \Longrightarrow \quad H \cong F(T) \text{ for some set } T.$$

This is surprising.

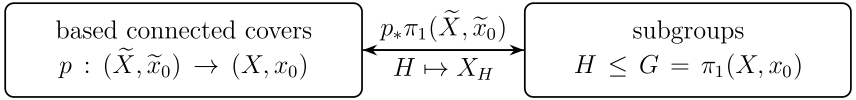
Subgroups of free abelian groups are free abelian, but their rank cannot increase.

For free groups, rank can increase dramatically.

Covering Spaces and Subgroups

Let X be a connected graph with basepoint x_0, and set

$$G = \pi_1(X, x_0).$$



Covering spaces give a Galois correspondence:
connected covers on one side, subgroups on the other.

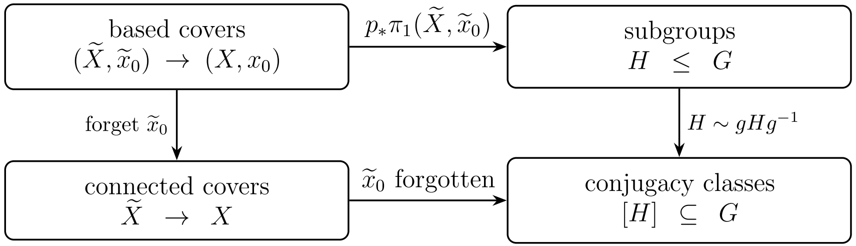
TikZ source: `slides/diagrams/covering-subgroup-correspondence.tex`

- A based connected cover determines a subgroup.
- The subgroup is the image of the induced map on π_1.
- Conversely, each subgroup builds a based connected cover.

Forgetting the Basepoint

The based correspondence is sharp:

$$(\widetilde{X}, \widetilde{x}_0) \longleftrightarrow H \leq G.$$

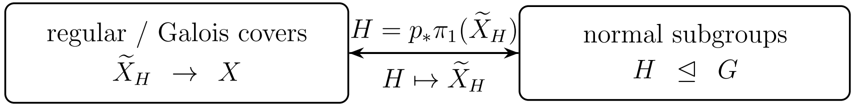


TikZ source: `slides/diagrams/covering-galois-correspondence.tex`

- If we forget $\widetilde{x}_0$, the subgroup is only determined up to conjugacy.
- So unbased connected covers correspond to conjugacy classes of subgroups.
- This is the same pattern as field-theoretic Galois correspondence: geometry on one side, subgroups on the other.

Normal Subgroups and Galois Covers

Some covers have maximal symmetry.



$$\mathrm{Deck}(\widetilde{X}_H/X) \cong G/H$$

TikZ source: `slides/diagrams/covering-normal-correspondence.tex`

- $H \trianglelefteq G$ corresponds to a regular, or Galois, covering.
- The quotient group G/H becomes the group of deck transformations.
- Regular covers are the topological analogue of normal field extensions.

Nielsen-Schreier from the Dictionary

Let

$$X = \bigvee^{n} S^1, \qquad \pi_1(X) \cong F_n.$$

Given $H \leq F_n$, the correspondence gives

$$X_H \longrightarrow X.$$

Since X is a graph, X_H is again a graph.

Thus

$$\pi_1(X_H) \cong F(T)$$

for some set T.

But the correspondence says this group maps to H:

$$\boxed{H \cong \pi_1(X_H) \cong F(T).}$$

Rank Is Not Monotone

In abelian groups:

$$H \leq \mathbb{Z}^n \quad \Rightarrow \quad H \cong \mathbb{Z}^r, \quad r \leq n.$$

In free groups:

$$H \leq F_n$$

does not imply $\mathrm{rank}(H) \leq n$.

There are subgroups of F_2 of rank $3, 4, 5, \ldots$, and even infinite rank.

Part VIII

Mathematicians

Walther von Dyck (1856-1934)

German mathematician.

Von Dyck helped make the language of **generators and relations** systematic.

Mathematical connection:

$$\langle S \mid R \rangle = F(S)/\langle\langle R \rangle\rangle.$$

Classroom message:

> A group can be described by symbols and equations among those symbols.

Max Dehn (1878-1952)

German mathematician, student of Hilbert.

Dehn introduced major algorithmic problems in group theory, including the **word problem**.

Question:

> Given a word in the generators, can we decide whether it represents the identity?

For free groups, the answer is easy: reduce the word.

For general finitely presented groups, the problem can be very hard.

Jakob Nielsen (1890-1959)

Danish mathematician.

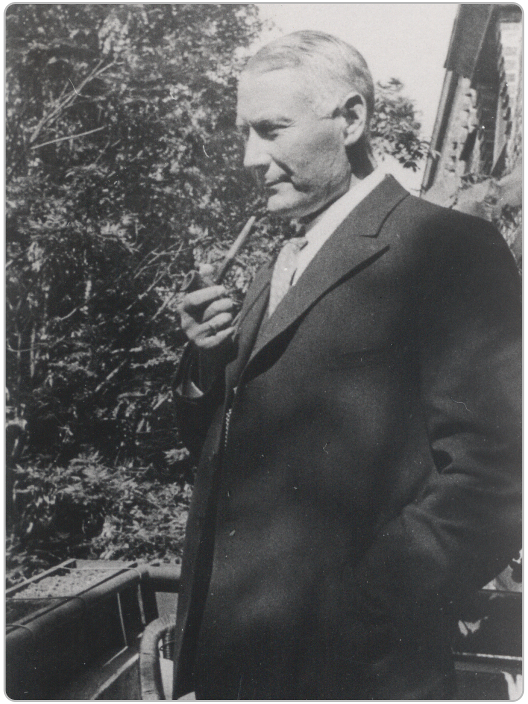
Nielsen studied transformations of free groups and surfaces.

His name appears in:

- Nielsen transformations;
- Nielsen reduction;
- Nielsen-Schreier theory.

Mathematical connection:

> Free groups have flexible generators, and changing generators is itself a rich structure.



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Otto Schreier (1901-1929)

Austrian mathematician.

Schreier gave foundational work on subgroups of free groups and covering-space methods.

The theorem carrying his name:

$$H \leq F(S) \Rightarrow H \text{ is free.}$$

Photo: MacTutor via Wikimedia Commons, public domain.

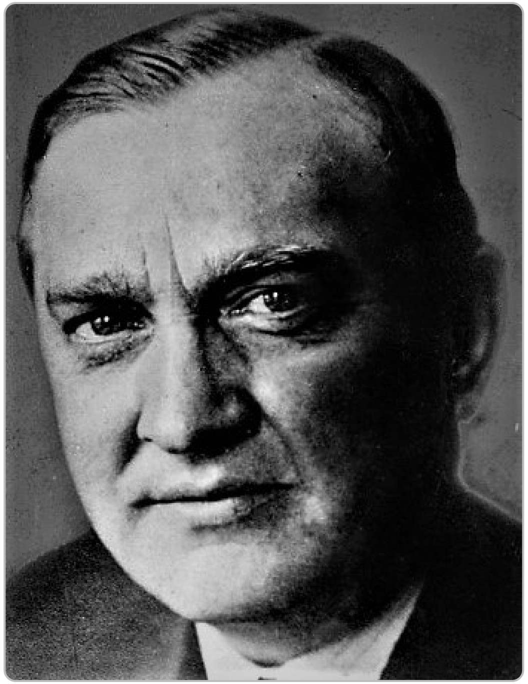
Stefan Banach (1892-1945)

Polish mathematician and one of the founders of functional analysis.

Banach spaces are named after him.

He was a central figure of the Lwow School of Mathematics.

In this lecture, his name appears through the Banach-Tarski paradox, where abstract group actions produce astonishing geometric consequences.



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Alfred Tarski (1901-1983)

Polish-American logician and mathematician.

Tarski was one of the major figures in model theory and mathematical logic.

He worked on truth, definability, decision problems, and algebraic logic.

In Banach-Tarski, his presence reminds us that the theorem is not just geometry: it is also about which sets are allowed to exist.

Banach-Tarski and Free Groups

Banach-Tarski paradox. A solid ball in $\mathbb{R}^3$ can be decomposed into finitely many pieces and rearranged by rigid motions to form two balls congruent to the original.

This is not a physical cutting theorem.

The pieces are highly non-measurable, and the proof uses the axiom of choice.

One algebraic engine behind the theorem:

$$F_2 \leq \mathrm{SO}(3).$$

Some rotations generate a free subgroup of the rotation group.

Concrete Rotations in $\mathrm{SO}(3)$

Let $\theta = \arccos(1/3)$ and set $s = 2\sqrt{2}/3$. Consider

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/3 & -s \\ 0 & s & 1/3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1/3 & -s & 0 \\ s & 1/3 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A rotates around the x-axis, and B rotates around the z-axis.

Both have determinant 1, so $A, B \in \mathrm{SO}(3)$.

The key theorem, proved by tracking reduced words, is:

$$\langle A, B \rangle \cong F_2.$$

That means no nontrivial reduced word in $A^{\pm 1}, B^{\pm 1}$ becomes the identity matrix.

How the Freeness Is Detected

The proof does not say that all rotations are free.

It uses this special angle $\cos\theta = 1/3$.

For a nontrivial reduced word W in $A^{\pm 1}, B^{\pm 1}$, after a harmless conjugation one may assume W ends in $A^{\pm 1}$.

Then induction on word length shows:

$$W(1,0,0) = \frac{1}{3^m}\left(p,\ q\sqrt{2},\ r\right), \qquad p,q,r \in \mathbb{Z}, \quad 3 \nmid q.$$

So $W(1,0,0) \neq (1,0,0)$.

Therefore $W \neq I$.

Cayley Graph of F_2

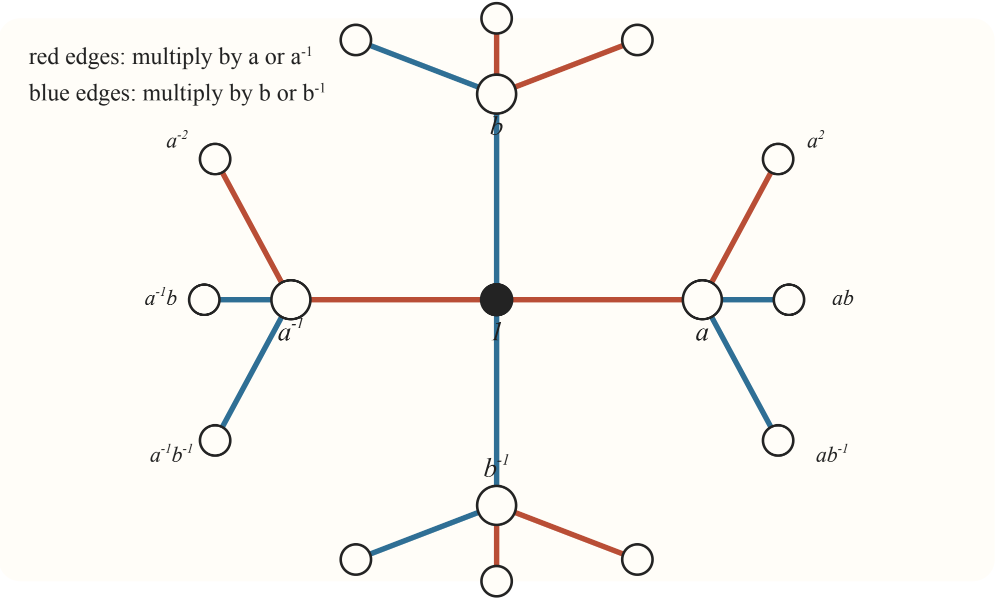
Let

$$F_2 = F(a, b).$$

- vertices are reduced words;
- edges multiply by $a^{\pm 1}$ or $b^{\pm 1}$;
- the graph is a 4-valent tree.

No cycles means:

$$\text{no nontrivial reduced word equals 1.}$$



Splitting F_2 by First Letter

Let W_x be the set of nonempty reduced words whose first letter is x.

Then:

$$F_2 \setminus \{1\} = W_a \sqcup W_{a^{-1}} \sqcup W_b \sqcup W_{b^{-1}}.$$

The tree structure gives two copies of the whole group:

$$F_2 = W_a \sqcup aW_{a^{-1}}, \qquad F_2 = W_b \sqcup bW_{b^{-1}}.$$

Example:

$$a \cdot (a^{-1}u) = u.$$

So left multiplication by a moves the a^{-1}-branch onto everything outside the a-branch.

This is the algebraic paradox inside Banach-Tarski.

From F_2 to the Sphere

Choose rotations $A, B \in \mathrm{SO}(3)$ such that

$$\langle A, B \rangle \cong F_2.$$

Remove the exceptional points fixed by nontrivial words.

On the remaining set $X \subset S^2$, the action is free.

Use choice to select one representative from each orbit:

$$E = \{x_O\}.$$

Each orbit is a copy of F_2:

$$X = \bigsqcup_{x \in E} F_2 x.$$

Define $X_a = W_a E$, $X_{a^{-1}} = W_{a^{-1}} E$, and similarly for b.

Then:

$$X = X_a \sqcup A X_{a^{-1}},$$

$$X = X_b \sqcup B X_{b^{-1}}.$$

The rotations perform the same moves as left multiplication in F_2.

Choice and non-measurable sets enter when these orbit pieces are assembled.

Summary

What to Remember

1. $F(S)$ is defined by a universal property:

$$\mathrm{Hom}_{\mathbf{Grp}}(F(S), G) \cong \mathrm{Func}(S, G).$$

2. Reduced words give a concrete normal form.
3. Abelianization kills commutators, and

$$F(S)^{ab} \cong F^{ab}(S).$$

4. Rank is the cardinality of a basis; abelianization shows it is well-defined.
5. Presentations add relation checks; D_n and Coxeter groups are built from them.
6. Graphs and covers explain why subgroups of free groups are free.
7. Banach-Tarski transports the paradoxical decomposition of F_2 through rotations.