MAT205: Abstract Algebra II

Subnormal Series, Solvable and Nilpotent Groups

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Goal

We have used normal subgroups to make quotient groups.

Today we use quotient groups repeatedly:

$$G = G_0 \rhd G_1 \rhd G_2 \rhd \cdots \rhd G_n = 1.$$

Question:

$$\boxed{\text{What do the successive quotients } G_i/G_{i+1} \text{ tell us about } G?}$$

Topics:

1. subnormal series;
2. composition series;
3. solvable groups;
4. nilpotent groups.

Examples Used

Group	Fact used
$\mathbb{Z}_n$	finite cyclic abelian group
S_3	solvable, not nilpotent
D_4	non-abelian nilpotent group of order 8
A_5	non-abelian simple group

Part I

Extensions and Subnormal Series

Normal Subgroup and Extension

If $N \triangleleft G$, then

$$1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1.$$

This is a short exact sequence:

$$\boxed{G \text{ is an extension of } G/N \text{ by } N.}$$

The groups N and G/N do not determine G up to isomorphism.

Example:

$$1 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 1, \qquad 1 \to \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2 \to 1.$$

The two middle groups are not isomorphic.

Example: S_3

The sign map gives

$$1 \longrightarrow A_3 \longrightarrow S_3 \xrightarrow{\text{sgn}} \{\pm 1\} \longrightarrow 1.$$

The kernel and quotient are

$$A_3 \cong \mathbb{Z}_3, \qquad S_3/A_3 \cong \mathbb{Z}_2.$$

Therefore

$$1 \lhd A_3 \lhd S_3 \quad \text{has abelian quotients.}$$

Subnormal Series

Definition. A **subnormal series** of G is a chain

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1$$

such that

$$G_{i+1} \triangleleft G_i \qquad \text{for } 0 \leq i < n.$$

The factor groups are

$$G_0/G_1, \quad G_1/G_2, \quad \ldots, \quad G_{n-1}/G_n.$$

Normality is only required **one step at a time**.

Normal Series

A **normal series** is a stronger kind of subnormal series:

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1$$

where each $G_i \triangleleft G$, not only $G_i \triangleleft G_{i-1}$.

So:

$$\text{normal series} \implies \text{subnormal series.}$$

Composition series are defined using subnormal series because normality is not transitive.

Normality Is Not Transitive

Let $D_4 = \langle r, s \mid r^4 = s^2 = 1,\ srs^{-1} = r^{-1} \rangle$, the symmetry group of a square.

Set

$$K = \langle r^2, s \rangle = \{1, r^2, s, r^2 s\}, \qquad H = \langle s \rangle.$$

Then

$$H \triangleleft K, \qquad K \triangleleft D_4, \qquad H \ntriangleleft D_4.$$

Thus

$$D_4 \supset K \supset H$$

is a valid one-step-at-a-time normal chain, even though H is not normal in D_4.

This is why a subnormal series only asks for one-step normality.

Refinement

A series can be made longer by inserting extra subgroups.

Example in $\mathbb{Z}_{12}$:

$$0 < \langle 3 \rangle < \mathbb{Z}_{12}$$

has factors

$$\mathbb{Z}_4, \qquad \mathbb{Z}_3.$$

Since $\mathbb{Z}_4$ still has a nontrivial proper subgroup, refine the chain:

$$0 < \langle 6 \rangle < \langle 3 \rangle < \mathbb{Z}_{12}.$$

Now every factor is cyclic of prime order:

$$\mathbb{Z}_2, \quad \mathbb{Z}_2, \quad \mathbb{Z}_3.$$

Part II

Composition Series

Simple Groups

Definition. A nontrivial group S is **simple** if its only normal subgroups are

$$1 \quad \text{and} \quad S.$$

Examples:

$$\mathbb{Z}_p \quad (p \text{ prime})$$

is simple.

A_5 is simple and non-abelian.

For $n \geq 5$, A_n is simple and non-abelian.

Composition Series

Definition. A **composition series** is a subnormal series

$$G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_n = 1$$

whose factors

$$G_i/G_{i+1}$$

are all simple.

The number n is the **composition length**.

Every finite group has a composition series.

Example: S_3

Use the chain

$$1 \triangleleft A_3 \triangleleft S_3.$$

The factors are

$$A_3/1 \cong \mathbb{Z}_3, \qquad S_3/A_3 \cong \mathbb{Z}_2.$$

Both are simple, so this is a composition series.

Composition factors:

$$\mathbb{Z}_3, \quad \mathbb{Z}_2.$$

Example: $\mathbb{Z}_{12}$

One composition series is

$$0 < \langle 6 \rangle < \langle 3 \rangle < \mathbb{Z}_{12}.$$

The factor orders are

$$2, \quad 2, \quad 3.$$

Another series can put the prime factors in a different order.

Jordan-Holder says the unordered list is intrinsic:

$$12 = 2 \cdot 2 \cdot 3.$$

Non-Example: $(\mathbb{Q}, +)$

The additive group $(\mathbb{Q}, +)$ is **divisible**:

$$\forall q \in \mathbb{Q},\ \forall n \geq 1, \quad \exists x \in \mathbb{Q} \text{ such that } nx = q.$$

Every quotient of a divisible abelian group is divisible.

$$\mathbb{Q}/H \text{ is divisible for every subgroup } H \leq \mathbb{Q}.$$

If $\mathbb{Q}$ had a composition series, the first factor

$$\mathbb{Q}/G_1$$

would be simple and abelian.

But every simple abelian group is $\mathbb{Z}_p$, and $\mathbb{Z}_p$ is not divisible.

Hence $(\mathbb{Q}, +)$ has no finite composition series.

Direct Product Calculation

Suppose

$$G = G_0 \geq G_1 \geq \cdots \geq G_m = 1, \qquad H = H_0 \geq H_1 \geq \cdots \geq H_n = 1$$

are composition series.

Then $G \times H$ has the composition series

$$G_0 \times H \geq G_1 \times H \geq \cdots \geq G_m \times H = 1 \times H_0 \geq 1 \times H_1 \geq \cdots \geq 1 \times H_n.$$

The factors are

$$\frac{G_i \times H}{G_{i+1} \times H} \cong \frac{G_i}{G_{i+1}}, \qquad \frac{1 \times H_j}{1 \times H_{j+1}} \cong \frac{H_j}{H_{j+1}}.$$

So the composition factors of $G \times H$ are the composition factors of G and H, with multiplicity.

Example: $S_3 \times \mathbb{Z}_{12}$

Use

$$S_3 \geq A_3 \geq 1, \qquad \mathbb{Z}_{12} \geq \langle 3 \rangle \geq \langle 6 \rangle \geq 0.$$

Then

$$S_3 \times \mathbb{Z}_{12} \geq A_3 \times \mathbb{Z}_{12} \geq 1 \times \mathbb{Z}_{12} \geq 1 \times \langle 3 \rangle \geq 1 \times \langle 6 \rangle \geq 1.$$

The factors are

$$\mathbb{Z}_2, \quad \mathbb{Z}_3, \quad \mathbb{Z}_3, \quad \mathbb{Z}_2, \quad \mathbb{Z}_2.$$

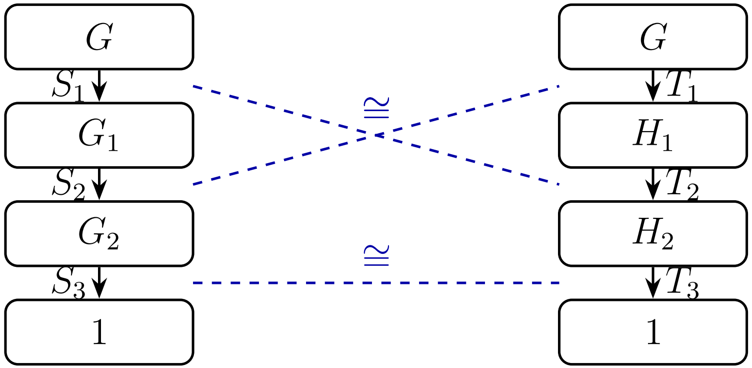
Thus the composition length is 5.

Jordan-Holder Theorem

Theorem. If a group G has two composition series, then:

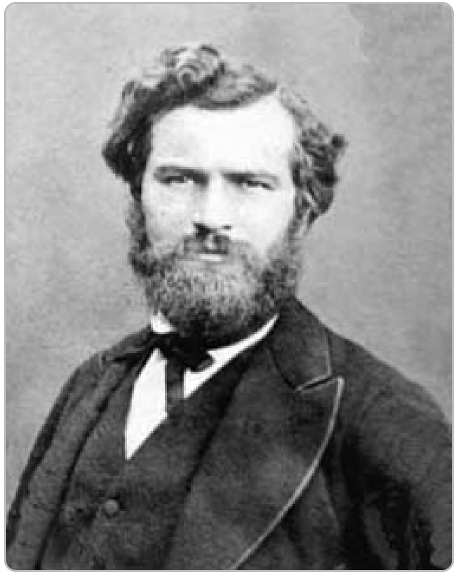
- the two series have the same length;
- after reordering, their simple factors are isomorphic.

So the composition factors are well-defined up to order.



Jordan–Holder: the simple factors agree up to order and isomorphism.

Camille Jordan and Otto Holder



Camille Jordan (1838-1922)

Developed early structure theory of finite groups and permutation groups.

Otto Holder (1859-1937)

Proved the modern uniqueness statement for composition factors.

Photos: Wikimedia Commons.

Consequences of Jordan-Holder

Jordan-Holder is not saying the subgroup chain is unique.

It says the composition factors are unique up to isomorphism and order.

The composition length is also independent of the chosen composition series.

The invariant is

$$\boxed{\{\, G_i/G_{i+1} \,\} \text{ as a multiset of isomorphism classes.}}$$

Composition Factors Do Not Determine G

Composition factors do not determine the isomorphism type of G.

Groups	Same composition factors
$\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2, \mathbb{Z}_2$
$\mathbb{Z}_6$ and S_3	$\mathbb{Z}_3, \mathbb{Z}_2$

Both rows give non-isomorphic groups with the same composition factors.

The extension structure is not determined by the composition factors.

The next definition imposes a condition on the factor groups:

$$\boxed{G_i/G_{i+1} \text{ abelian for all } i.}$$

Part III

Solvable Groups

Solvable Groups

Definition.

G is **solvable** if there is a subnormal series

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1$$

with $G_{i+1} \lhd G_i$ and

$$G_i/G_{i+1} \text{ abelian for } 0 \leq i < n.$$

This is the definition used in the Galois theorem on solvability by radicals.

Abelian Factors and Commutators

For a subnormal series

$$G = G_0 \geq G_1 \geq \cdots \geq G_n = 1,$$

the factor G_i/G_{i+1} is abelian iff

$$G_i/G_{i+1} \text{ abelian} \iff [G_i, G_i] \leq G_{i+1}.$$

Thus a subnormal series has abelian factors iff

$$[G_i, G_i] \leq G_{i+1} \quad \text{for all } i.$$

This is the local commutator condition for solvability.

Derived Series

Define

$$G^{(0)} = G, \qquad G^{(r+1)} = [G^{(r)}, G^{(r)}].$$

Then

$$G^{(r)}/G^{(r+1)}$$

is abelian for every r.

The series

$$G = G^{(0)} \geq G^{(1)} \geq G^{(2)} \geq \cdots$$

is the **derived series**.

Derived-Series Criterion

Theorem.

$$G \text{ is solvable} \iff G^{(m)} = 1 \text{ for some } m.$$

The least such m is the **derived length** of G.

This criterion is equivalent to the existence of a subnormal series with abelian factors.

For finite groups, this criterion can be checked by computing commutator subgroups.

Example: S_3

In S_3,

$$[S_3, S_3] = A_3.$$

Since A_3 is abelian,

$$[A_3, A_3] = 1.$$

So

$$S_3 \triangleright A_3 \triangleright 1$$

is the derived series.

Therefore S_3 is solvable of derived length 2.

Example: A_5

A_5 is simple and non-abelian.

Its derived subgroup is normal:

$$[A_5, A_5] \lhd A_5.$$

Since A_5 is non-abelian, this subgroup is not 1.

Since A_5 is simple, it must be all of A_5:

$$[A_5, A_5] = A_5.$$

So the derived series never shrinks. Hence A_5 is not solvable.

Composition-Factor Test

Theorem. For finite groups:

$$\boxed{G \text{ is solvable} \iff \text{all composition factors are cyclic of prime order.}}$$

Proof idea:

simple abelian groups are exactly

$$\mathbb{Z}_p.$$

A finite group is not solvable iff at least one composition factor is non-abelian simple.

Solvable by Radicals

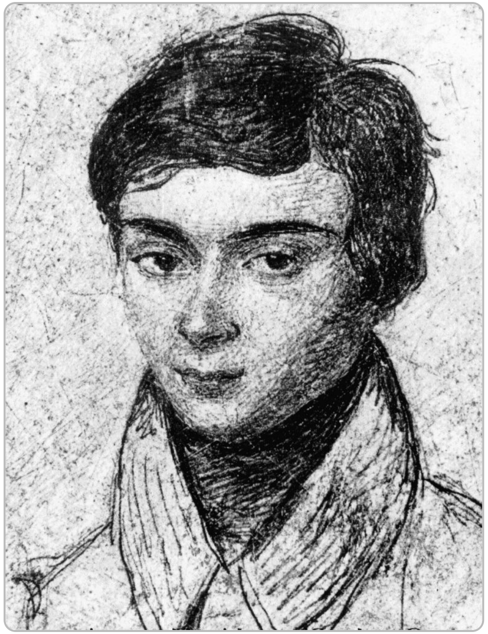
Let $f \in \mathbb{Q}[x]$, and let L be its splitting field over $\mathbb{Q}$.

Theorem (Galois). f is solvable by radicals over $\mathbb{Q}$ iff

$$\mathrm{Gal}(L/\mathbb{Q})$$

is a solvable group.

Here "solvable group" means the abelian-factor definition above.



Evariste Galois. Wikimedia Commons.

Radical Extensions

Assume K contains the n-th roots of unity.

Let

$$L = K(\alpha), \qquad \alpha^n = a \in K,$$

and assume L/K is Galois.

For $\sigma \in \mathrm{Gal}(L/K)$,

$$\sigma(\alpha)^n = a, \qquad \sigma(\alpha) = \zeta\alpha \quad (\zeta \in \mu_n).$$

Thus

$$\mathrm{Gal}(L/K) \hookrightarrow \mu_n,$$

so $\mathrm{Gal}(L/K)$ is abelian.

Generic Polynomial

The polynomial

$$x^n + t_1 x^{n-1} + \cdots + t_n \in \mathbb{Q}(t_1, \ldots, t_n)[x]$$

has Galois group S_n over $\mathbb{Q}(t_1, \ldots, t_n)$.

Also:

$$S_n \text{ is solvable for } n \leq 4, \qquad S_n \text{ is not solvable for } n \geq 5.$$

Therefore the generic degree n polynomial is solvable by radicals for $n \leq 4$ and not solvable by radicals for $n \geq 5$.

Burnside's Solvability Theorem

William Burnside. Wikimedia Commons.

Theorem (Burnside). If

$$|G| = p^a q^b$$

for primes p, q, then G is solvable.

By the composition-factor test, every composition factor of G is cyclic of prime order.

In particular, no non-abelian simple group has order $p^a q^b$.

Part IV

Central Series and Nilpotent Groups

Center

The center is

$$Z(G) = \{z \in G : zg = gz \text{ for all } g \in G\}.$$

It is a normal subgroup:

$$Z(G) \triangleleft G.$$

The center measures the elements that already commute with all of G:

$$z \in Z(G) \iff [z, g] = 1 \text{ for all } g \in G.$$

Upper Central Series

Define inductively:

$$Z_0(G) = 1.$$

For $i \geq 0$, define $Z_{i+1}(G)$ by

$$Z_{i+1}(G)/Z_i(G) = Z\big(G/Z_i(G)\big).$$

This gives an ascending chain

$$1 = Z_0(G) \triangleleft Z_1(G) \triangleleft Z_2(G) \triangleleft \cdots .$$

In particular, $Z_1(G) = Z(G)$.

Lower Central Series

Define inductively:

$$\gamma_1(G) = G, \qquad \gamma_{i+1}(G) = [\gamma_i(G), G].$$

This gives

$$G = \gamma_1(G) \triangleright \gamma_2(G) \triangleright \gamma_3(G) \triangleright \cdots .$$

This is the **lower central series**.

Central Series

A **central series** is a chain

$$G = G_0 \geq G_1 \geq \cdots \geq G_c = 1$$

such that

$$[G, G_i] \leq G_{i+1} \qquad (0 \leq i < c).$$

Equivalently,

$$G_i/G_{i+1} \leq Z(G/G_{i+1}).$$

Nilpotent Groups

Definition. G is **nilpotent** if

$$Z_c(G) = G$$

for some $c \geq 0$.

Equivalently,

$$\gamma_{c+1}(G) = 1$$

for some $c \geq 0$.

Equivalently, G has a central series.

The least such c is the **nilpotency class** of G.

Abelian Groups

If G is abelian, then

$$Z(G) = G.$$

So every nontrivial abelian group is nilpotent of class 1.

Therefore:

$$\boxed{\text{abelian} \implies \text{nilpotent.}}$$

Example: D_4

For the square group

$$D_4 = \langle r, s \mid r^4 = s^2 = 1,\ srs^{-1} = r^{-1} \rangle,$$

The center is $Z(D_4) = \{1, r^2\}$.

Since $D_4/Z(D_4) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$,

$$[D_4, D_4] \le Z(D_4).$$

Also $[D_4, Z(D_4)] = 1$, so

$$D_4 > Z(D_4) > 1$$

is a central series.

D_4 is nilpotent of class 2.

Unitriangular Groups

Let $U_n(\mathbb{F}_p)$ be the group of upper triangular matrices with 1 on the diagonal.

Example:

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let U_n^k be the subgroup whose first $k-1$ superdiagonals are zero.

One has

$$[U_n^a, U_n^b] \leq U_n^{a+b}.$$

Thus

$$U_n(\mathbb{F}_p) \text{ is nilpotent of class } n-1.$$

Finite p-Groups Are Nilpotent

We proved earlier:

$$G \text{ a finite } p\text{-group} \quad \Longrightarrow \quad Z(G) \neq 1.$$

Then apply the same fact to

$$G/Z(G).$$

Induction on $|G|$ gives $Z_c(G) = G$ for some c.

Therefore:

$$\boxed{\text{Every finite } p\text{-group is nilpotent.}}$$

Finite Nilpotent Groups and Sylow Theory

For a finite group G, the following are equivalent:

- G is nilpotent.
- Every Sylow subgroup of G is normal.
- G is the direct product of its Sylow subgroups:

$$G \cong \prod_{p \mid |G|} P_p.$$

Here P_p denotes the Sylow p-subgroup of G.

Nilpotent Implies Solvable

The lower central series is

$$\gamma_1(G) = G, \qquad \gamma_{i+1}(G) = [\gamma_i(G), G].$$

The derived series satisfies

$$G^{(m)} \leq \gamma_{2^m}(G).$$

So if $\gamma_{c+1}(G) = 1$, then $G^{(m)} = 1$ once $2^m > c$.

Thus:

$$\boxed{\text{nilpotent} \implies \text{solvable.}}$$

Converse Fails

S_3 is solvable:

$$1 \triangleleft A_3 \triangleleft S_3, \qquad A_3 \cong \mathbb{Z}_3, \quad S_3/A_3 \cong \mathbb{Z}_2.$$

S_3 is not nilpotent.

For finite nilpotent groups, every Sylow subgroup is normal.

The Sylow 2-subgroups of S_3 have order 2, and

$$n_2 = 3.$$

Hence no Sylow 2-subgroup is normal.

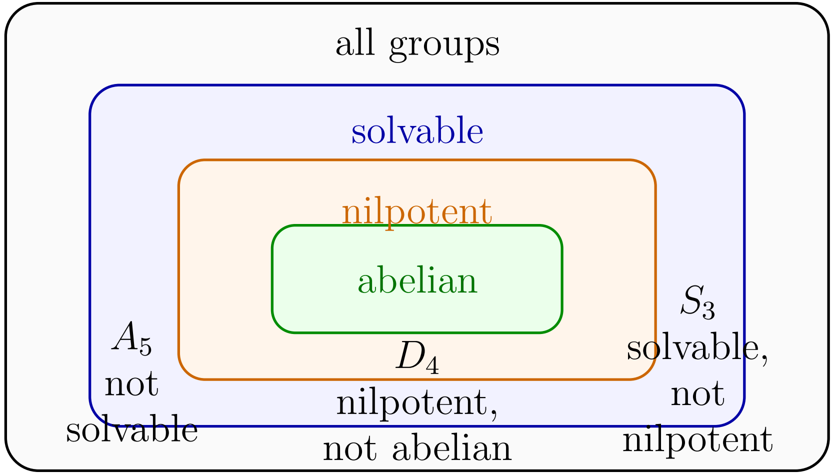
Part V

Summary Criteria

Implication Diagram

The inclusions are strict:

- D_4 is nilpotent but not abelian.
- S_3 is solvable but not nilpotent.
- A_5 is not solvable.



Example Table

Group	Composition factors	Solvable?	Nilpotent?
$\mathbb{Z}_n$	$\mathbb{Z}_p$ with multiplicity $v_p(n)$	yes	yes
D_4	three copies of $\mathbb{Z}_2$	yes	yes
S_3	$\mathbb{Z}_3, \mathbb{Z}_2$	yes	no
A_5	A_5	no	no

A_5 is non-abelian simple.

Criteria for Solvability

For a finite group G, any of the following proves solvability:

- Find a subnormal series with abelian factors.
- Compute the derived series and show it reaches 1.
- Show all composition factors are cyclic of prime order.
- Use known closure properties: subgroups, quotients, extensions of solvable groups are solvable.

To prove non-solvability, it is enough to exhibit a non-abelian simple composition factor.

Criteria for Nilpotence

For a finite group G, any of the following proves nilpotence:

- Build the upper central series until it reaches G.
- Build the lower central series until it reaches 1.
- If G is finite, check whether every Sylow subgroup is normal.
- If G is a finite p-group, conclude immediately that it is nilpotent.

To prove non-nilpotence, it is enough to find a non-normal Sylow subgroup.

Comparison

Notion	Criterion for finite groups
composition factors	simple factors in a composition series
solvable	all composition factors are $\mathbb{Z}_p$
nilpotent	all Sylow subgroups are normal

The implications are

$$\text{abelian} \implies \text{nilpotent} \implies \text{solvable.}$$

The reverse implications are false:

$$D_4 \text{ is nilpotent but not abelian,} \qquad S_3 \text{ is solvable but not nilpotent.}$$

False Statements

1. "A subnormal series must be a normal series."

False: subnormal only requires $G_{i+1} \triangleleft G_i$.

2. "Composition factors determine the group."

False: $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have the same composition factors.

3. "Solvable implies nilpotent."

False: S_3 is solvable but not nilpotent.

Summary

What to Remember

1. A subnormal series decomposes a group into factor groups G_i/G_{i+1}.
2. A composition series is a fully refined subnormal series; Jordan-Holder says its simple factors are well-defined up to order.
3. For $f \in \mathbb{Q}[x]$, solvability by radicals is equivalent to solvability of its Galois group.
4. For finite groups, solvable iff all composition factors are cyclic groups $\mathbb{Z}_p$.
5. The derived series is the standard criterion for solvability in examples.
6. G is nilpotent iff $Z_c(G) = G$ for some c, equivalently iff $\gamma_{c+1}(G) = 1$ for some c.
7. A finite group is nilpotent iff it is the direct product of its Sylow subgroups.

The implication chain is

$$\text{abelian} \implies \text{nilpotent} \implies \text{solvable} \implies \text{group}.$$