XIAMEN UNIVERSITY MALAYSIA

Academic Session: 2026/04

MAT205 Abstract Algebra II

Midterm Test Marking Scheme

28^{th} May 2026 (Thursday)

16:00 - 17:45 (1 hour and 45 minutes)

Student Name	
Student ID	

Instructions

1. This test consists of THREE (3) questions. Answer ALL questions.
2. This test contributes 20% to the overall score for the course.
3. This test covers Weeks 1-6.
4. You are allowed to use any result mentioned in the lectures and Fraleigh's textbook (state the textbook version when quoting a result) without proof.
5. You are NOT allowed to communicate with any human being via any method throughout the duration of the test, except your lecturer and tutor. Raise your hand quietly if you have any questions during the exam. Your lecturer or tutor will attend to you shortly.

Problem	**Marks**	**Out of**
Question 1		**30**
Question 2		**40**
Question 3		**30**
Total		**100**

Question 1. (30 marks) Determine whether the following statements are true or false. If the statement is true, give a short explanation or proof. If the statement is false, explain why by giving a counterexample.

(a) [5 marks] Every subgroup of index 2 is normal in the group.

Solution. True. Let $H \leq G$ and $[G : H] = 2$. Then there are only two left cosets and only two right cosets. If $g \notin H$, then both gH and Hg are equal to $G \setminus H$. Hence $gH = Hg$ for all $g \in G$, so $H \trianglelefteq G$.

(b) [5 marks] Let $H \trianglelefteq G$ and $K \trianglelefteq H$. Then $K \trianglelefteq G$.

Solution. False. Take $G = A_4$, let

$$H = V_4 = \{e, (12)(34), (13)(24), (14)(23)\},$$

and let $K = \langle (12)(34) \rangle$. Then $H \trianglelefteq A_4$ and $K \trianglelefteq H$ because H is abelian. However K is not normal in A_4, since conjugation by elements of A_4 permutes the three nonidentity elements of V_4.

(c) [5 marks] Every group of order p^2, where p is prime, is abelian.

Solution. True. If $|G| = p^2$, then G is a finite p-group, so $Z(G) \neq 1$. Thus $|Z(G)| = p$ or p^2. If $|Z(G)| = p$, then $G/Z(G)$ has order p and is cyclic, which implies G is abelian, a contradiction to $Z(G) \neq G$. Hence $Z(G) = G$ and G is abelian.

(d) [5 marks] The free group $F(a, b)$ on two generators is abelian.

Solution. False. For example, by the universal property of $F(a, b)$, there is a homomorphism $F(a, b) \to S_3$ sending $a \mapsto (12)$ and $b \mapsto (23)$. Since $(12)(23) \neq (23)(12)$ in S_3, the elements ab and ba cannot be equal in $F(a, b)$.

(e) [5 marks] $\mathbb{Z}_4 \times \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$.

Solution. True. Since $\mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_3$, we have

$$\mathbb{Z}_4 \times \mathbb{Z}_6 \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3.$$

Also $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$, so

$$\mathbb{Z}_2 \times \mathbb{Z}_{12} \cong \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_3.$$

Thus the two groups are isomorphic.

(f) [5 marks] If H is characteristic in K and $K \trianglelefteq G$, then $H \trianglelefteq G$.

Solution. True. For any $g \in G$, conjugation by g sends K to itself because $K \trianglelefteq G$. Therefore conjugation by g restricts to an automorphism of K. Since H is characteristic in K, this automorphism preserves H. Hence $gHg^{-1} = H$ for all $g \in G$, so $H \trianglelefteq G$.

Question 2. (40 marks) Solve the following questions.

(a) [10 marks] Let $\gamma = (1234) \in S_4$. Determine the centralizer

$$C_{S_4}(\gamma) = \{\sigma \in S_4 \mid \sigma\gamma\sigma^{-1} = \gamma\}.$$

Hence determine the size of the conjugacy class of γ in S_4.

Solution. If $\sigma\gamma\sigma^{-1} = \gamma$, then $\sigma\gamma = \gamma\sigma$. Thus once $\sigma(1)$ is known, the values of σ on all other elements are forced:

$$\sigma(2) = \sigma(\gamma(1)) = \gamma(\sigma(1)), \quad \sigma(3) = \gamma^2(\sigma(1)), \quad \sigma(4) = \gamma^3(\sigma(1)).$$

Hence σ must be a power of γ. Therefore

$$C_{S_4}(\gamma) = \langle \gamma \rangle = \{e, \gamma, \gamma^2, \gamma^3\}.$$

By the orbit-stabilizer theorem for the conjugation action,

$$|[\gamma]| = \frac{|S_4|}{|C_{S_4}(\gamma)|} = \frac{24}{4} = 6.$$

(b) [5 marks] For every integer $n \geq 1$, show that

$$n\mathbb{Z}/n^2\mathbb{Z} \cong \mathbb{Z}_n.$$

Solution. Define $\phi : n\mathbb{Z} \to \mathbb{Z}_n$ by $\phi(nk) = \overline{k}$. This is a well-defined surjective homomorphism. Its kernel consists of all nk such that $k \equiv 0 \pmod{n}$, so $\ker(\phi) = n^2\mathbb{Z}$. By the first isomorphism theorem,

$$n\mathbb{Z}/n^2\mathbb{Z} \cong \mathbb{Z}_n.$$

(c) [10 marks] Write down all abelian groups of order 72 up to isomorphism.

Solution. Since $72 = 2^3 \cdot 3^2$, the 2-primary part is one of

$$\mathbb{Z}_8, \qquad \mathbb{Z}_4 \times \mathbb{Z}_2, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,$$

and the 3-primary part is one of

$$\mathbb{Z}_9, \qquad \mathbb{Z}_3 \times \mathbb{Z}_3.$$

Thus the abelian groups of order 72, up to isomorphism, are

$$\mathbb{Z}_8 \times \mathbb{Z}_9, \qquad \mathbb{Z}_8 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \qquad \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_9,$$
$$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9, \qquad \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3.$$

(d) [5 marks] Determine the composition factors of

$$A = \mathbb{Z}_{12} \times \mathbb{Z}_{18}.$$

Solution. Since A is finite abelian, all its composition factors are cyclic of prime order. Also

$$|A| = 12 \cdot 18 = 216 = 2^3 \cdot 3^3.$$

Hence the composition factors, counted with multiplicity, are

$$\mathbb{Z}_2,\ \mathbb{Z}_2,\ \mathbb{Z}_2,\ \mathbb{Z}_3,\ \mathbb{Z}_3,\ \mathbb{Z}_3.$$

(e) [5 marks] Let G be a group of order 45. Show that G is not simple.

Solution. We have $45 = 3^2 \cdot 5$. Let n_5 be the number of Sylow 5-subgroups of G. By Sylow's third theorem,

$$n_5 \mid 9, \qquad n_5 \equiv 1 \pmod{5}.$$

The only divisor of 9 congruent to 1 modulo 5 is 1. Thus G has a unique Sylow 5-subgroup, which is normal. It is proper and nontrivial, so G is not simple.

(f) [5 marks] Give an example of a solvable group which is not nilpotent. Justify your answer.

Solution. The group S_3 is solvable but not nilpotent. It is solvable because

$$S_3' = A_3, \qquad A_3' = 1,$$

so the derived series reaches the trivial group. It is not nilpotent because, in a finite nilpotent group, every Sylow subgroup is normal. The Sylow 2-subgroups of S_3 have order 2 and are not normal.

Question 3. (30 marks) Prove the following statements.

(a) [10 marks] Let H be a subgroup of a group G. Show that the centralizer of H in G is normal in the normalizer of H in G, i.e.

$$Z_G(H) \trianglelefteq N_G(H).$$

Solution. Consider the conjugation action of $N_G(H)$ on H. This gives a group homomorphism

$$\varphi : N_G(H) \to \mathrm{Aut}(H), \qquad \varphi(g)(h) = ghg^{-1}.$$

The map is well-defined because $g \in N_G(H)$ implies $gHg^{-1} = H$. Its kernel is

$$\ker(\varphi) = \{g \in N_G(H) \mid ghg^{-1} = h \text{ for all } h \in H\}.$$

This is exactly $Z_G(H)$. Since the kernel of a homomorphism is normal in the domain, $Z_G(H) \trianglelefteq N_G(H)$.

(b) [10 marks] Let G be a group. Show that both $Z(G)$ and $[G, G]$ are characteristic subgroups of G.

Solution. Let $\alpha \in \mathrm{Aut}(G)$.

First, if $z \in Z(G)$ and $g \in G$, write $g = \alpha(x)$ for some $x \in G$. Then

$$\alpha(z)g = \alpha(z)\alpha(x) = \alpha(zx) = \alpha(xz) = \alpha(x)\alpha(z) = g\alpha(z).$$

Hence $\alpha(z) \in Z(G)$, so $\alpha(Z(G)) \subseteq Z(G)$. Applying the same argument to α^{-1} gives equality. Thus $Z(G)$ is characteristic.

Second, α sends commutators to commutators:

$$\alpha([x, y]) = [\alpha(x), \alpha(y)].$$

Hence $\alpha([G, G]) \subseteq [G, G]$. Again applying the same argument to α^{-1} gives equality. Therefore $[G, G]$ is characteristic in G.

(c) [10 marks] Let G be a group of order 21.

i. Show that G has a normal Sylow 7-subgroup P.

ii. Let Q be a Sylow 3-subgroup of G. Show that $G \cong P \rtimes Q$.

iii. Determine how many isomorphism classes of groups of order 21 there are.

Solution. Let n_7 be the number of Sylow 7-subgroups. By Sylow's third theorem,

$$n_7 \mid 3, \qquad n_7 \equiv 1 \pmod 7.$$

Hence $n_7 = 1$. Thus the Sylow 7-subgroup P is normal in G.

Let Q be a Sylow 3-subgroup. Then $|P| = 7$, $|Q| = 3$, and $P \cap Q = 1$. Since P is normal,

$$|PQ| = \frac{|P||Q|}{|P \cap Q|} = 21,$$

so $PQ = G$. Therefore G is the semidirect product $P \rtimes Q$.

Since $P \cong \mathbb{Z}_7$ and $Q \cong \mathbb{Z}_3$, the semidirect product is determined by a homomorphism

$$\varphi : \mathbb{Z}_3 \to \mathrm{Aut}(\mathbb{Z}_7).$$

Now $\mathrm{Aut}(\mathbb{Z}_7) \cong \mathbb{Z}_6$. There is one trivial homomorphism, giving the cyclic group

$$\mathbb{Z}_7 \times \mathbb{Z}_3 \cong \mathbb{Z}_{21}.$$

There are also nontrivial homomorphisms whose image is the unique subgroup of order 3 in $\mathrm{Aut}(\mathbb{Z}_7)$. These give one non-abelian isomorphism class: for example,

$$\langle a, b \mid a^7 = b^3 = e,\ bab^{-1} = a^2 \rangle.$$

The two possible nontrivial actions differ by replacing b with b^{-1}, so they define isomorphic groups. Hence there are exactly two isomorphism classes of groups of order 21.

END OF MARKING SCHEME