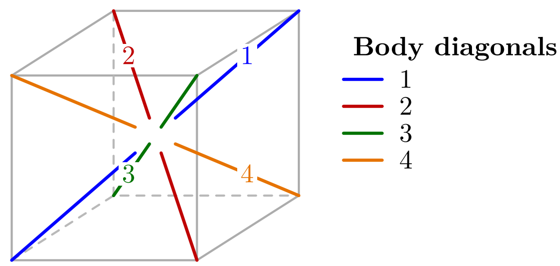
MAT205 Assignment 1

Covers Lectures 1–8. This assignment has two long problems. Each problem starts from one important object and asks you to read its algebraic structure step by step.

Problem 1. The Rotation Group of a Cube

Let G be the group of orientation-preserving rotations of a cube.



the four body diagonals

a. The cube has four body diagonals, labeled $1, 2, 3, 4$ in the figure. Show that every rotation of the cube permutes these four diagonals. Hence define a homomorphism

$$\varphi : G \to S_4.$$

b. Prove that φ is injective.

Hint: if a rotation fixes all four body diagonals, then it fixes the cube.

c. Prove that φ is surjective by showing that every transposition $(ij) \in S_4$ lies in $\operatorname{im}\varphi$.

Hint: put the cube at the vertices $(\pm1, \pm1, \pm1)$, and label the body diagonals by

$$D_1 = \{\pm(1,1,1)\}, \quad D_2 = \{\pm(-1,1,1)\}, \quad D_3 = \{\pm(-1,-1,1)\}, \quad D_4 = \{\pm(1,-1,1)\}.$$

Use half-turns around axes through the midpoints of suitable pairs of opposite edges.

Conclude that

$$G \cong S_4.$$

d. Use Sylow theory to determine the possible values of n_3 and n_2, the numbers of Sylow 3- and Sylow 2-subgroups of G. Then use the geometry of the cube, or the identification $G \cong S_4$, to determine the actual values of n_3 and n_2.

e. Explain geometrically what one Sylow 3-subgroup represents. Explain geometrically what one Sylow 2-subgroup represents.

f. Put the cube at the vertices $(\pm1, \pm1, \pm1)$, and consider the four vertices

$$T = \{(1,1,1),\ (-1,-1,1),\ (-1,1,-1),\ (1,-1,-1)\}.$$

Show that T is a regular tetrahedron inscribed in the cube. Let

$$K = \{g \in G : g(T) = T\}.$$

Prove that, under the isomorphism $G \cong S_4$, the subgroup K corresponds to A_4.

Now let $V = \mathbb{R}^3$, with the cube centered at the origin, and define

$$V^K = \{v \in V : k(v) = v \text{ for every } k \in K\}.$$

Compute V^K. In your answer, distinguish between preserving the tetrahedron T setwise and fixing vectors in V pointwise.

g. Let $V_4 = \{1, (12)(34), (13)(24), (14)(23)\} \le S_4$. Use the chain

$$1 \triangleleft C_2 \triangleleft V_4 \triangleleft A_4 \triangleleft S_4$$

to find the composition factors of G.

$$\begin{array}{c} S_4 \\ \big\downarrow \mathbb{Z}_2 \\ A_4 \\ \big\downarrow \mathbb{Z}_3 \\ V_4 \\ \big\downarrow \mathbb{Z}_2 \\ C_2 \\ \big\downarrow \mathbb{Z}_2 \\ 1 \end{array}$$

h. Decide whether G is solvable. Decide whether G is nilpotent. Justify both answers using the criteria from the lectures.

Problem 2. Automorphisms of Finite Abelian Groups

Let p be a prime. The first nontrivial example is

$$A = \mathbb{Z}_{p^2} \oplus \mathbb{Z}_p.$$

Write $A = \langle e_1 \rangle \oplus \langle e_2 \rangle$, where $|e_1| = p^2$ and $|e_2| = p$.

$$\begin{array}{ccc} A & \overset{\text{quotient}}{\dashrightarrow} & A/pA \cong \mathbb{Z}_p^2 \\ \big\downarrow & & \\ A[p] = \{a \in A : pa = 0\} \cong \mathbb{Z}_p^2 & & \\ \big\downarrow & & \\ pA \cong \mathbb{Z}_p & & \\ \big\downarrow & & \\ 0 & & \end{array}$$

a. Compute pA, $A[p]$, and A/pA explicitly in terms of e_1, e_2.

b. Prove that pA and $A[p]$ are characteristic subgroups of A. Explain why every automorphism of A must preserve the filtration

$$0 \subset pA \subset A[p] \subset A.$$

c. Show that every endomorphism $f : A \to A$ has the form

$$f(e_1) = ae_1 + be_2, \qquad f(e_2) = pce_1 + de_2,$$

where $a \in \mathbb{Z}/p^2\mathbb{Z}$ and $b, c, d \in \mathbb{F}_p$.

d. Prove that f is an automorphism if and only if

$$a \in (\mathbb{Z}/p^2\mathbb{Z})^\times, \qquad d \in \mathbb{F}_p^\times.$$

Using columns for the images of e_1, e_2, the automorphisms are the matrices

$$\begin{pmatrix} a & pc \\ b & d \end{pmatrix}, \qquad a \in (\mathbb{Z}/p^2\mathbb{Z})^\times, \quad b, c \in \mathbb{F}_p, \quad d \in \mathbb{F}_p^\times.$$

e. Deduce that

$$|\operatorname{Aut}(A)| = p^3(p-1)^2.$$

f. Now let

$$B = \mathbb{Z}_{p^{\lambda_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{\lambda_r}}, \qquad \lambda_1 \geq \cdots \geq \lambda_r.$$

Explain why every subgroup $p^k B$ is characteristic. State the general principle suggested by parts (a)–(e): an automorphism of a finite abelian p-group is an invertible endomorphism whose matrix entries respect the p-power filtration.

g. Finally let M be any finite abelian group with primary decomposition

$$M = \bigoplus_p M_p, \qquad M_p = \{m \in M : p^k m = 0 \text{ for some } k \geq 1\}.$$

Prove that each M_p is characteristic and conclude that

$$\operatorname{Aut}(M) \cong \prod_p \operatorname{Aut}(M_p).$$