MAT205 Quiz 2

Time: 30 minutes. Total: 15 marks.

This quiz develops a single running example: the Galois theory of the splitting field of $x^5 - 3$ over $\mathbb{Q}$. Question 1 checks a basic fact about $\mathbb{Q}(\sqrt[5]{3})$; Questions 2 and 3 build up the full Galois group and its correspondence. Give concise answers, but include enough explanation to justify your conclusions.

Name: ____

Student ID: __________________________________

Questions

Question 1. True or false with justification [3 marks]

For each statement, decide whether it is true or false. If it is true, give a short reason. If it is false, give a counterexample.

a. [1 mark] $[\mathbb{Q}(\sqrt[5]{3}) : \mathbb{Q}] = 5$.

b. [1 mark] Every algebraic extension of $\mathbb{Q}$ has finite degree over $\mathbb{Q}$.

c. [1 mark] $\mathbb{Q}(\sqrt[5]{3})$ is a normal extension of $\mathbb{Q}$.

Question 2. Solve [6 marks]

Let F be the splitting field of $x^5 - 3$ over $\mathbb{Q}$, and write $\beta = \sqrt[5]{3}$ (the real fifth root) and $\omega = e^{2\pi i/5}$, so $F = \mathbb{Q}(\beta, \omega)$. Define $\sigma, \tau \in \mathrm{Gal}(F/\mathbb{Q})$ by

$$\sigma : \ \beta \mapsto \beta\omega, \quad \omega \mapsto \omega, \qquad\qquad \tau : \ \beta \mapsto \beta, \quad \omega \mapsto \omega^2.$$

a. [4 marks] Verify that σ, τ are well-defined automorphisms of F with σ of order 5 and τ of order 4, and that $\tau\sigma\tau^{-1} = \sigma^2$. Using this, show $[F : \mathbb{Q}] = 20$, conclude that $\mathrm{Gal}(F/\mathbb{Q}) = \langle \sigma, \tau \rangle$, and identify the isomorphism type of the group.

b. [2 marks] State the fixed field of $\langle\sigma\rangle$ and the fixed field of $\langle\tau\rangle$ (no proof required here — you will use this in Question 3).

Question 3. Prove [6 marks]

With F, σ, τ as in Question 2:

a. [2 marks] Prove that $\langle\tau\rangle$ is not a normal subgroup of $\mathrm{Gal}(F/\mathbb{Q})$.

b. [4 marks] Using Sylow's theorem and part (a), determine exactly how many subgroups of $\mathrm{Gal}(F/\mathbb{Q})$ are conjugate to $\langle\tau\rangle$, and write down the fixed field of each one.